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Curvature theory of generalized second order gauge connections

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Abstract. Lately a big attention has been paid to second order gauge connections, but the investigation are mostly restricted to the d-connection. Here this connection is generalized. The curvature tensors and the Ricci equations are obtained, which for the generalized connection have a rather simple form.

1. Adapted basis in TF

Let F be an $n+m+l$ dimensional C^∞ manifold. Some point $v \in F$ in the local charts (U,φ) and (U',φ') has coordinates (x^i, y^a, z^p) and $(x^{i'}, y^{a'}, z^{p'})$ respectively. In $U \cap U'$ the allowable coordinate transformations are given by the equations

$$
\begin{aligned}
&x^{i'} = x^{i'}(x) && i,j,h,k = \overline{1,n},\\
(1.1)\qquad &y^{a'} = y^{a'}(x,y) && a,b,c,d,e = \overline{n+1,n+m},\\
&z^{p'} = z^{p'}(x,z) && p,q,r,s,t = \overline{n+m+1,n+m+l},
\end{aligned}
$$

where

$$(1.2)\qquad \operatorname{rank}\left[\frac{\partial x^{i'}}{\partial x^i}\right] = n,\ \operatorname{rank}\left[\frac{\partial y^{a'}}{\partial y^a}\right] = m,\ \operatorname{rank}\left[\frac{\partial z^{p'}}{\partial z^p}\right] = l.$$

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From (1.1) and (1.2) it follows that inverse transformations of the form

$$x^i = x^i(x'),\ y^a = y^a(x',y'),\ z^p = z^p(x',z') \tag{1.1$'$}$$

exist.

Proposition 1.1. *The coordinate transformations of type* (1.1) *form a group.*

The tangent space TF is spanned at any point $v \in F$ by $n+m+l$ basis vectors, which form the basis

$$\bar{B} = \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial z^p} \right\}. \tag{1.3}$$

The elements of $\bar{B}$ with respect to (1.1) are transformed in the following way:

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial}{\partial x^{i'}} + \frac{\partial y^{a'}}{\partial x^i}\,\frac{\partial}{\partial y^{a'}} + \frac{\partial z^{p'}}{\partial x^i}\,\frac{\partial}{\partial z^{p'}}, \tag{1.4}$$

$$\frac{\partial}{\partial y^a} = \frac{\partial y^{a'}}{\partial y^a}\,\frac{\partial}{\partial y^{a'}}, \qquad \frac{\partial}{\partial z^p} = \frac{\partial z^{p'}}{\partial z^p}\,\frac{\partial}{\partial z^{p'}}. \tag{1.5}$$

The functions $\mathcal{N}_{i'}^{b'}(x',y')$ and $\mathcal{M}_{i'}^{p'}(x',z')$ are the coefficients of non-linear connections of the second order if they satisfy the following law of transformation:

(1.6)

$$\text{(a)}\quad \mathcal{N}_i^b(x,y) = \mathcal{N}_{i'}^{b'}(x',y')\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial y^b}{\partial y^{b'}} + \frac{\partial y^{a'}}{\partial x^i}\,\frac{\partial y^b}{\partial y^{a'}},$$

$$\text{(b)}\quad \mathcal{M}_i^p(x,z) = \mathcal{M}_{i'}^{p'}(x',z')\frac{\partial x^{i'}}{\partial x^i}\,\frac{\partial z^p}{\partial z^{p'}} + \frac{\partial z^{p'}}{\partial x^i}\,\frac{\partial z^p}{\partial z^{p'}}.$$

Remark. In many papers as in [11], [13], [14] $\mathcal{N}$ and $\mathcal{M}$ are interchanged. The reason for this change is the fact that for $n+m$ dimensional F, $\mathcal{N}$ is reduced to N used before as in [6]–[10].

Using the notation

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{N}_i^b(x,y)\frac{\partial}{\partial y^b} - \mathcal{M}_i^p(x,z)\frac{\partial}{\partial z^p} \tag{1.7}$$

from (1.6) and (1.7) we obtain

$$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i}\,\frac{\delta}{\delta x^{i'}}. \tag{1.8}$$

Thus the adapted basis

$$B(\mathcal{N},\mathcal{M})=\left\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^a},\frac{\partial}{\partial z^p}\right\} \tag{1.9}$$

of TF is obtained. It is clear that as many adapted bases can be formed, as functions $\mathcal{N}$ and $\mathcal{M}$ satisfying (1.6)(a) and (1.6)(b) can be found. The transformations of the elements of $B(\mathcal{N},\mathcal{M})$, T_HF, $T_{V_1}F$, $T_{V_2}F$ are determined by (1.5) and (1.8).

Let us denote by T_HF, $T_{V_1}F$, $T_{V_2}F$ the subspaces of TF spanned by $\{\frac{\delta}{\delta x^i}\}$, $\{\frac{\partial}{\partial y^a}\}$, $\{\frac{\partial}{\partial z^p}\}$ respectively, then

$$TF=T_HF\oplus T_{V_1}F\oplus T_{V_2}F. \tag{1.10}$$

Any vector field X in TF can be written in the form

$$X=X^i\frac{\delta}{\delta x^i}+X^a\frac{\partial}{\partial y^a}+X^p\frac{\partial}{\partial z^p}. \tag{1.11}$$

The coordinates of X under (1.1) are transformed in the following way:

$$X^{i'}=X^i\frac{\partial x^{i'}}{\partial x^i},\ X^{a'}=X^a\frac{\partial y^{a'}}{\partial y^a},\ X^{p'}=X^p\frac{\partial z^{p'}}{\partial z^p}. \tag{1.12}$$

The adapted basis of T^*F is

$$B^*(\mathcal{N},\mathcal{M})=\{dx^i,\delta y^a,\delta z^p\}, \tag{1.13}$$

where

$$\delta y^a=dy^a+\mathcal{N}^a_i(x,y)dx^i \tag{1.14}$$

$$\delta z^p=dz^p+\mathcal{M}^p_i(x,z)dx^i. \tag{1.15}$$

The elements of B^* are transformed in the following way:

$$dx^i=\frac{\partial x^i}{\partial x^{i'}}dx^{i'},\ \delta y^a=\frac{\partial y^a}{\partial y^{a'}}\delta y^{a'},\ \delta z^p=\frac{\partial z^p}{\partial z^{p'}}\delta z^{p'}. \tag{1.16}$$

Let us denote the subspaces of T^*F spanned by $\{dx^i\}$, $\{\delta y^a\}$, $\{\delta z^p\}$ respectively by T^*_HF, $T^*_{V_1}F$, $T^*_{V_2}F$, then

$$T^*F=T^*_HF\oplus T^*_{V_1}F\oplus T^*_{V_2}F. \tag{1.17}$$

Any 1-form field $\omega\in T^*F$ can be written in the form:

$$\omega=\omega_i dx^i+\omega_a\delta y^a+\omega_p\delta z^p. \tag{1.18}$$

The coordinates of the 1-form ω with respect to (1.1) are transformed in the following way:

$$\omega_i = \omega_{i'} \frac{\partial x^{i'}}{\partial x^i}, \ \omega_a = \omega_{a'} \frac{\partial y^{a'}}{\partial y^a}, \ \omega_p = \omega_{p'} \frac{\partial z^{p'}}{\partial z^p}. \tag{1.19}$$

2. Gauge covariant derivatives of the second order

Let $\nabla : TF \times TF \to TF$ ($\times$ is the Descartes product) be a linear connection, such that $\nabla : (X, Y) \to \nabla_X Y \in TF$, $\forall X, Y \in TF$. The operator ∇ is called generalized gauge connection of the second order. It is called d-gauge connection of the second order if $\nabla_X Y$ is in $T_H F$, $T_{V_1} F$ or $T_{V_2} F$ provided Y is in $T_H F$, $T_{V_1} F$ or $T_{V_2} F$ respectively, $\forall X \in TF$. It has been studied by many authors, mostly romanian geometers.

Here we shall not make this restriction on ∇. In the following we shall use the abbreviations: $\delta_k = \frac{\delta}{\delta x^k}$, $\partial_k = \frac{\partial}{\partial x^k}$, $\partial_a = \frac{\partial}{\partial y^a}$, $\partial_p = \frac{\partial}{\partial z^p}$.

Definition 2.1. The generalized gauge connection ∇ of the second order is defined by

$$\begin{aligned} &\text{(a)} \quad \nabla_{\delta_i} \delta_\beta = F_{\beta\ i}^{\ \kappa} \delta_\kappa, \\ &\text{(b)} \quad \nabla_{\partial_a} \delta_\beta = C_{\beta\ a}^{\ \kappa} \delta_\kappa \\ &\text{(c)} \quad \nabla_{\partial_p} \delta_\beta = L_{\beta\ p}^{\ \kappa} \delta_\kappa, \end{aligned} \tag{2.1}$$

where $\beta = j$ or $\beta = b$ or $\beta = q$ and

$$T_{\dots}^{\cdot\cdot\kappa} \delta_\kappa = T_{\dots}^{\cdot\cdot k} \delta_k + T_{\dots}^{\cdot\cdot c} \partial_c + T_{\dots}^{\cdot\cdot r} \partial_r. \tag{2.2}$$

We shall use the abbreviated form of (2.1):

$$\nabla_{\delta_\alpha} \delta_\beta = \Gamma_{\beta\ \alpha}^{\ \kappa} \delta_\kappa. \tag{2.3}$$

From (2.2) and (2.3) it follows:

If $\alpha = i$, then $\Gamma = F$; if $\alpha = a$, then $\Gamma = C$; if $\alpha = p$, then $\Gamma = L$.

Proposition 2.1. *If X is the vector field* (1.11) *defined on F, then the following equations are valid:*

$$\begin{aligned} &\nabla_{\delta_i} X = X^\alpha_{\ |i} \delta_\alpha, \quad X^\alpha_{\ |i} = \delta_1 X^\alpha + F_\beta{}^\alpha{}_i X^\beta \\ &\nabla_{\partial_a} X = X^\alpha|_a \delta_\alpha, \quad X^\alpha|_a = \partial_a X^\alpha + C_\beta{}^\alpha{}_a X^\beta \\ &\nabla_{\partial_p} X = X^\alpha\|_p \delta_\alpha, \quad X^\alpha\|_p = \partial_p X^\alpha + L_\beta{}^\alpha{}_p X^\beta, \end{aligned} \tag{2.4}$$

where

$$\Gamma_{\dot{\beta}\ .}X^{\beta} = \Gamma_{\dot{j}\ .}X^{j} + \Gamma_{\dot{b}\ .}X^{b} + \Gamma_{\dot{q}\ .}X^{q}. \tag{2.5}$$

Theorem 2.1. *If X and Y are vector fields in TF expressed in the basis B, and ∇ the second order gauge connection defined by (2.1), then the following equation is valid:*

$$\nabla_Y X = (X^{\alpha}_{\ |\beta})Y^{\beta}\delta_{\alpha}, \tag{2.6}$$

where

$$\ ^{\cdots}_{\cdots|\beta}Y^{\beta} = \ ^{\cdots}_{\cdots|j}Y^{j} + \ ^{\cdots}_{\cdots}|_{b}Y^{b} + \ ^{\cdots}_{\cdots}\|_{q}Y^{q}. \tag{2.7}$$

Theorem 2.2. *All covariant derivatives $X^{\alpha}_{\ |i}$, $X^{\alpha}|_{a}$, $X^{\alpha}\|_{p}$ ($\alpha = j$, or $\alpha = b$, or $\alpha = p$) from (2.4) are transformed as tensors with respect to (1.1) if all connection coefficients from (2.1) are transformed as tensors, except the following which have the form:*

$$F_{j\ i}^{\ k} = F_{j'\ i'}^{\ k'}\frac{\partial x^{i'}}{\partial x^{i}}\ \frac{\partial x^{k}}{\partial x^{k'}}\ \frac{\partial x^{j'}}{\partial x^{j}} + \frac{\partial^2 x^{k'}}{\partial x^{i}\partial x^{j}}\ \frac{\partial x^{k}}{\partial x^{k'}} \tag{2.8}$$

$$F_{b\ i}^{\ c} = F_{b'\ i'}^{\ c'}\frac{\partial x^{i'}}{\partial x^{i}}\ \frac{\partial y^{b'}}{\partial y^{b}}\ \frac{\partial y^{c}}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial x^{i}\partial y^{b}}\ \frac{\partial y^{c}}{\partial y^{c'}} - \mathcal{N}_{i}^{a}\frac{\partial^2 y^{c'}}{\partial y^{b}\partial y^{a}}\ \frac{\partial y^{c}}{\partial y^{c'}} \tag{2.9}$$

$$F_{q\ i}^{\ r} = F_{q'\ i'}^{\ r'}\frac{\partial x^{i'}}{\partial x^{i}}\ \frac{\partial z^{q'}}{\partial z^{q}}\ \frac{\partial z^{r}}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial x^{i}\partial z^{q}}\ \frac{\partial z^{r}}{\partial z^{r'}} - \mathcal{M}_{i}^{s}\frac{\partial^2 z^{r'}}{\partial z^{s}\partial z^{q}}\ \frac{\partial z^{r}}{\partial z^{r'}} \tag{2.10}$$

$$C_{b\ a}^{\ c} = C_{b'\ a'}^{\ c'}\frac{\partial y^{b'}}{\partial y^{b}}\ \frac{\partial y^{a'}}{\partial y^{a}}\ \frac{\partial y^{c}}{\partial y^{c'}} + \frac{\partial^2 y^{c'}}{\partial y^{a}\partial y^{b}}\ \frac{\partial y^{c}}{\partial y^{c'}} \tag{2.11}$$

$$L_{q\ p}^{\ r} = L_{q'\ p'}^{\ r'}\frac{\partial z^{q'}}{\partial z^{q}}\ \frac{\partial z^{p'}}{\partial z^{p}}\ \frac{\partial z^{r}}{\partial z^{r'}} + \frac{\partial^2 z^{r'}}{\partial z^{q}\partial z^{p}}\ \frac{\partial z^{r}}{\partial z^{r'}} \tag{2.12}$$

The proof is given in [7]

Theorem 2.3. *The torsion tensor T for the second order gauge connection ∇ has the form:*

$$T(X,Y) = T_{\alpha\ \beta}^{\ \kappa}Y^{\alpha}X^{\beta}\delta_{\kappa}, \tag{2.13}$$

where

$$T_{\alpha\ \beta}^{\ \kappa} = \Gamma_{\alpha\ \beta}^{\ \kappa} - \Gamma_{\beta\ \alpha}^{\ \kappa} \tag{2.14}$$

except the following components:

$$(2.15)\quad \begin{aligned} &(a)\quad T_{j\ i}^{\ c} = F_{j\ i}^{\ c} - F_{i\ j}^{\ c} - \mathcal{N}_{i\ j}^{\ c}\\ &(b)\quad T_{j\ b}^{\ c} = C_{j\ b}^{\ c} - F_{b\ j}^{\ c} + \partial_b \mathcal{N}_j^c = -T_{b\ j}^{\ c}\\ &(c)\quad T_{j\ i}^{\ r} = F_{j\ i}^{\ r} - F_{i\ j}^{\ r} - \mathcal{M}_{i\ j}^{\ q}\\ &(d)\quad T_{p\ i}^{\ r} = F_{p\ i}^{\ r} - L_{i\ p}^{\ r} - \partial_p \mathcal{M}_i^r = -T_{p\ i}^{\ r}, \end{aligned}$$

where

$$(2.16)\quad \begin{aligned} &(a)\quad \mathcal{N}_{i\ j}^{\ c} = (\partial_j \mathcal{N}_i^c - \mathcal{N}_j^b \partial_b \mathcal{N}_i^c) - (i,j)\\ &(b)\quad \mathcal{M}_{i\ j}^{\ q} = (\partial_j \mathcal{M}_i^q - \mathcal{M}_j^p \partial_p \mathcal{M}_i^q) - (i,j) \end{aligned}$$

The proof is given in [7].

3. The curvature theory of ∇

The curvature tensor for the second order gauge connection ∇ is defined as usual

$$(3.1)\qquad R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

If we use the notation

$$X = X^\alpha \partial_\alpha,\ Y = Y^\beta \partial_\beta,\ Z = Z^\gamma \partial_\gamma,$$

then we have

$$(3.2)\quad \begin{aligned} \nabla_X \nabla_Y Z &= \nabla_{X^\alpha \partial_\alpha} \nabla_{Y^\beta \partial_\beta} Z^\gamma \partial_\gamma\\ &= \nabla_{X^\alpha \partial_\alpha} [Y^\beta (\partial_\beta Z^\gamma) \partial_\gamma + Y^\beta Z^\gamma \Gamma_{\gamma\ \beta}^{\ \kappa} \partial_\kappa]\\ &= X^\alpha (\partial_\alpha Y^\beta)(\partial_\beta Z^\gamma)\partial_\gamma + X^\alpha Y^\beta (\partial_\alpha \partial_\beta Z^\gamma)\partial_\gamma\\ &\quad + X^\alpha Y^\beta (\partial_\beta Z^\gamma) \Gamma_{\gamma\ \alpha}^{\ \kappa} \partial_\kappa + X^\alpha (\partial_\alpha Y^\beta) Z^\gamma \Gamma_{\gamma\ \beta}^{\ \kappa} \partial_\kappa\\ &\quad + X^\alpha Y^\beta (\partial_\alpha Z^\gamma) \Gamma_{\gamma\ \beta}^{\ \kappa} \partial_\kappa + X^\alpha Y^\beta Z^\gamma (\partial_\alpha \Gamma_{\gamma\ \beta}^{\ \kappa}) \partial_\kappa\\ &\quad + X^\alpha Y^\beta Z^\gamma \Gamma_{\gamma\ \beta}^{\ \theta} \Gamma_{\theta\ \alpha}^{\ \kappa} \partial_\kappa. \end{aligned}$$

From

$$(3.3)\quad [X,Y] = X^\alpha (\partial_\alpha Y^\beta)\partial_\beta - Y^\alpha (\partial_\alpha X^\beta)\partial_\beta + X^\alpha Y^\beta (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha),$$

where

$$(3.4)\quad \begin{aligned} &X^\alpha Y^\beta (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) = X^\alpha Y^\beta K_{\alpha\ \beta}^{\ \kappa} \partial_\kappa\\ &= X^i Y^j (\mathcal{N}_{i\ j}^{\ c} \partial_c + \mathcal{M}_{i\ j}^{\ q} \partial_q) + (X^i Y^b - Y^i X^b)(\partial_b \mathcal{N}_i^c)\partial_c\\ &\quad + (X^i Y^q - Y^i X^q)(\partial_q \mathcal{M}_i^p)\partial_p, \end{aligned}$$

it follows:

$$
\begin{aligned}
\nabla_{[XY]}Z = {} & X^\alpha(\partial_\alpha Y^\beta)(\partial_\beta Z^\gamma)\partial_\gamma + X^\alpha(\partial_\alpha Y^\beta)Z^\gamma\Gamma_{\gamma\ \alpha}^{\ \kappa}\partial_\kappa \\
& - Y^\alpha(\partial_\alpha X^\beta)(\partial_\beta Z^\gamma)\partial_\gamma - Y^\alpha(\partial_\alpha X^\beta)Z^\gamma\Gamma_{\gamma\ \beta}^{\ \kappa}\partial_\kappa \\
& + X^\alpha Y^\beta[(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)Z^\gamma]\partial_\gamma \\
& + X^\alpha Y^\beta Z^\gamma K_{\alpha\ \beta}^{\ \theta}\Gamma_{\gamma\ \theta}^{\ \kappa}\partial_\kappa.
\end{aligned}
\tag{3.5}
$$

Substituting (3.2) and (3.5) into (3.1) we obtain

$$
\begin{aligned}
R(X,Y)Z = {} & [K_{\gamma\ \beta\alpha}^{\ \kappa}X^\alpha Y^\beta - (\mathcal{N}_{i\ j}^{\ c}C_{\gamma\ c}^{\ \kappa} + \mathcal{M}_{i\ j}^{\ q}L_{\gamma\ q}^{\ \kappa})X^iY^j \\
& - \partial_b\mathcal{N}_i^c C_{\gamma\ c}^{\ \kappa}(X^iY^b - Y^iX^b) \\
& - \partial_q\mathcal{M}_i^p L_{\gamma\ p}^{\ \kappa}(X^iY^q - Y^iX^q)]Z^\gamma\partial_\kappa,
\end{aligned}
\tag{3.6}
$$

where

$$
K_{\gamma\ \beta\alpha}^{\ \kappa} = (\partial_\alpha\Gamma_{\gamma\ \beta}^{\ \kappa} - \Gamma_{\gamma\ \alpha}^{\ \theta}\Gamma_{\theta\ \beta}^{\ \kappa}) - (\alpha,\beta).
\tag{3.7}
$$

As the indices α, β, γ, κ belong to one of the sets $\{i,j,k,l,\dots\}$, $\{a,b,c,d,\dots\}$, $\{p,q,r,s,t,\dots\}$ (corresponding to T_HF, $T_{V_1}F$, $T_{V_2}F$ respectively), so on the TF we have $3^4 = 81$ types of curvature tensors. It is meaningless to introduce different letters as R, P, S for the curvature tensors as in Finsler geometry.

We shall denote

$$
R_{\gamma\ \beta\alpha}^{\ \kappa} = K_{\gamma\ \beta\alpha}^{\ \kappa}
\tag{3.8}
$$

for all (β,α) except when $(\beta,\alpha) = (j,i)$, $(\beta,\alpha) = (b,i)$, $(\beta,\alpha) = (i,b)$, $(\beta,\alpha) = (q,i)$ and $(\beta,\alpha) = (i,q)$. In these cases we have

$$
\begin{aligned}
R_{\gamma\ ji}^{\ \kappa} &= K_{\gamma\ ji}^{\ \kappa} - \mathcal{N}_{i\ j}^{\ c}C_{\gamma\ c}^{\ \kappa} - \mathcal{M}_{i\ j}^{\ q}L_{\gamma\ q}^{\ \kappa}, \\
R_{\gamma\ ib}^{\ \kappa} &= K_{\gamma\ ib}^{\ \kappa} + \partial_b\mathcal{N}_i^c C_{\gamma\ c}^{\ \kappa}, \\
R_{\gamma\ bi}^{\ \kappa} &= K_{\gamma\ bi}^{\ \kappa} - \partial_b\mathcal{N}_i^c C_{\gamma\ c}^{\ \kappa}, \\
R_{\gamma\ iq}^{\ \kappa} &= K_{\gamma\ iq}^{\ \kappa} + \partial_q\mathcal{M}_i^p L_{q\ p}^{\ \kappa}, \\
R_{\gamma\ qi}^{\ \kappa} &= K_{\gamma\ qi}^{\ \kappa} - \partial_q\mathcal{M}_i^p L_{\gamma\ p}^{\ \kappa}.
\end{aligned}
\tag{3.9}
$$

From (3.7), (3.8) and (3.9) it is obvious that

$$
R_{\gamma\ \beta\alpha}^{\ \kappa} = -R_{\gamma\ \alpha\beta}^{\ \kappa}.
\tag{3.10}
$$

We can write (3.6) in the form:

$$
\begin{aligned}
R(X,Y)Z = \Big[&\frac{1}{2}K_{\gamma\ \beta\alpha}^{\ \kappa}(X^\alpha Y^\beta - Y^\alpha X^\beta)\\
&-\frac{1}{2}(\mathcal{N}_{i\ j}^{\ c}C_{\gamma\ c}^{\ \kappa} + \mathcal{M}_{i\ j}^{\ q}L_{\gamma\ q}^{\ \kappa})(X^iY^j - Y^iX^j)\\
&-\frac{1}{2}\partial_b\mathcal{N}_i^c C_{\gamma\ c}^{\ \kappa}(X^iY^b - Y^iX^b)\\
&+\frac{1}{2}\partial_b\mathcal{N}_i^c C_{\gamma\ c}^{\ \kappa}(Y^iX^b - X^iY^b)\\
&-\frac{1}{2}\partial_q\mathcal{M}_i^p L_{\gamma\ p}^{\ \kappa}(X^iY^q - Y^iX^q)\\
&+\frac{1}{2}\partial_q\mathcal{M}_i^p L_{\gamma\ p}^{\ \kappa}(Y^iX^q - X^iY^q)\Big]Z^\gamma\delta_\kappa
\end{aligned}
\tag{3.11}
$$

For $(\beta,\alpha)=(j,i)$ the sum of the first and the second line in (3.11) is equal to $\frac{1}{2}R_{\gamma\ ji}^{\ \kappa}(X^iY^j - Y^iX^j)$, for $(\beta,\alpha)=(b,i)$ the sum of the first and the third line in (3.11) is equal to $\frac{1}{2}R_{\gamma\ bi}^{\ \kappa}(X^iY^b - Y^iX^b)$ etc.

From (3.8)–(3.11) follows

Theorem 3.1. *The curvature tensor of the second order gauge connection* ∇ *has the form*

$$
R(X,Y)Z = \frac{1}{2}R_{\gamma\ \beta\alpha}^{\ \kappa}(X^\alpha Y^\beta - Y^\alpha X^\beta)Z^\gamma\delta_\kappa, \tag{3.12}
$$

where the components of R *are determined by* (3.8) *and* (3.9).

Formula (3.12) is short and elegant, but the explicit form of the curvature tensor is much longer, for instance if $(\beta,\alpha)=(b,i)$ then from (3.7) and (3.9) we have:

$$
\begin{aligned}
R_{\gamma\ bi}^{\ \kappa} &= \delta_i C_{\gamma\ b}^{\ \kappa} - F_{\gamma\ i}^{\ \theta}C_{\theta\ b}^{\ \kappa} - \partial_b F_{\gamma\ i}^{\ \kappa} + C_{\gamma\ b}^{\ \theta}F_{\theta\ i}^{\ \kappa} - \partial_b\mathcal{N}_i^c C_{\gamma\ c}^{\ \kappa}\\
&= \delta_i C_{\gamma\ b}^{\ \kappa} - F_{\gamma\ i}^{\ k}C_{k\ b}^{\ \kappa} - F_{\gamma\ i}^{\ c}C_{c\ b}^{\ \kappa} - F_{\gamma\ i}^{\ r}C_{r\ b}^{\ \kappa}\\
&\quad - \partial_b F_{\gamma\ i}^{\ \kappa} + C_{\gamma\ b}^{\ k}F_{k\ i}^{\ \kappa} + C_{\gamma\ b}^{\ c}F_{c\ i}^{\ \kappa} + C_{\gamma\ b}^{\ r}F_{r\ i}^{\ \kappa} - \partial_b\mathcal{N}_i^c C_{\gamma\ c}^{\ \kappa}.
\end{aligned}
$$

4. Ricci identities for ∇

From (2.6) it follows

$$\nabla_X \nabla_Y Z = [(Z^{\gamma}_{\ |\beta})Y^{\beta}]_{|\alpha} X^{\alpha}\delta_{\gamma} = (Z^{\gamma}_{\ |\beta|\alpha} Y^{\beta} + Z^{\gamma}_{\ |\beta} Y^{\beta}_{\ |\alpha}) X^{\alpha}\delta_{\gamma}. \tag{4.1}$$

From (2.6), (3.3) and (3.4) we obtain

$$\nabla_{[X,Y]} Z = Z^{\gamma}_{\ |\beta}[X,Y]^{\beta}\delta_{\gamma} = A + B, \tag{4.2}$$

where

$$\begin{aligned} A &= Z^{\gamma}_{\ |\beta}[X^{\alpha}(\partial_{\alpha} Y^{\beta}) - Y^{\alpha}(\partial_{\alpha} X^{\beta})]\delta_{\gamma} \\ &= Z^{\gamma}_{\ |\beta}[X^{\alpha} Y^{\beta}_{\ |\alpha} - Y^{\alpha} X^{\beta}_{\ |\alpha} - (\Gamma_{\theta\ \alpha}^{\ \beta} - \Gamma_{\alpha\ \theta}^{\ \beta}) X^{\alpha} Y^{\theta}]\delta_{\gamma} \end{aligned} \tag{4.3}$$

$$\begin{aligned} B &= X^i Y^j [Z^{\gamma}|_c \mathcal{N}_{i\ j}^{\ c} + Z^{\gamma}\|_q \mathcal{M}_{i\ j}^{\ q}]\delta_{\gamma} \\ &+ (X^i Y^b - Y^i X^b) Z^{\gamma}|_c (\partial_b \mathcal{N}_i^c)\delta_{\gamma} \\ &+ (X^i Y^q - Y^i X^q) Z^{\gamma}\|_p (\partial_q \mathcal{M}_i^p)\delta_{\gamma}. \end{aligned} \tag{4.4}$$

Taking into account (2.14) and (2.15) we obtain

$$A + B = \left[Z^{\gamma}_{\ |\beta}\left(X^{\alpha} Y^{\beta}_{\ |\alpha} - Y^{\alpha} X^{\beta}_{\ |\alpha}\right) - Z^{\gamma}_{\ |\kappa} T_{\beta\ \alpha}^{\ \kappa} X^{\alpha} Y^{\beta}\right]\delta_{\gamma}. \tag{4.5}$$

From (4.1), (4.2) and (4.5) we obtain

$$\begin{aligned} R(X,Y)Z &= (Z^{\gamma}_{\ |\beta|\alpha} - Z_{|\alpha|\beta} + Z^{\gamma}_{\ |\kappa} T_{\beta\ \alpha}^{\ \kappa}) X^{\alpha} Y^{\beta}\delta_{\gamma} \\ &= \frac{1}{2}(Z^{\gamma}_{\ |\beta|\alpha} - Z^{\gamma}_{\ |\alpha|\beta} + Z^{\gamma}_{\ |\kappa} T_{\beta\ \alpha}^{\ \kappa})(X^{\alpha} Y^{\beta} - Y^{\alpha} X^{\beta})\delta_{\gamma}. \end{aligned} \tag{4.6}$$

From (4.6) and (3.12) it follows:

Theorem 4.1. *The Ricci equations for the second order gauge connection ∇ have the form:*

$$Z^{\gamma}_{\ |\beta|\alpha} - Z^{\gamma}_{\ |\alpha|\beta} + Z^{\gamma}_{\ |\kappa} T_{\beta\ \alpha}^{\ \kappa} = R_{\kappa\ \beta\alpha}^{\ \gamma} Z^{\kappa}. \tag{4.7}$$

(4.7) contains 3^3 types of Ricci equations, because each Greek index may be an element from one of the sets: $\{i,j,h,k,l\}$, $\{a,b,c,d,e\}$, $\{p,q,r,s,t\}$.

For $(\beta,\alpha) = (j,i)$ (4.7) becomes

$$\begin{aligned} Z^{\gamma}_{\ |j|i} - Z^{\gamma}_{\ |i|j} + Z^{\gamma}_{\ |k} T_{j\ i}^{\ k} + Z^{\gamma}|_c T_{j\ i}^{\ c} + Z^{\gamma}\|_p T_{j\ i}^{\ p} \\ = R_{k\ ji}^{\ \gamma} Z^k + R_{c\ ji}^{\ \gamma} Z^c + R_{p\ ji}^{\ \gamma} Z^p, \end{aligned}$$

for $(\beta,\alpha)=(p,i)$ (4.7) takes the form

$$Z^\gamma\|_{p|i} - Z^\gamma_{|i}\|_p + Z^\gamma_{|k}T_{p\ i}^{\ k} + Z^\gamma|_c T_{p\ i}^{\ c} + Z^\gamma\|_r T_{p\ i}^{\ r} = R^{\gamma}_{k\ pi}Z^k + R_{c\ pi}^{\ \gamma}Z^c + R_{r\ pi}^{\ \gamma}Z^r, \text{ e.t.c.}$$

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