Almost additive functions on semigroups and a functional equation

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§ 1. J. ACZÉL has asked in [1] what can be said about functions g satisfying the conditional functional equation

$$(1) \qquad g(x+y)g(x)g(y) \neq 0 \quad \text{implies} \quad \frac{1}{g(x+y)} = \frac{1}{g(x)} + \frac{1}{g(y)}.$$

Here g is assumed to be of the type: $S \to K$ where $(S, +)$ is a semigroup and $(K, +, \cdot)$ is a field (both not necessarily commutative). As was shown in [8], equation [1] may be reduced to the following one:

$$(2) \quad f(x+y) \neq 0 \quad \text{and} \quad f(x) \neq 0 \quad \text{and} \quad f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y).$$

This proves that, in genuine, Aczél's problem is of a (semi)group-theoretical nature. It was also pointed out in [8] that even in the case where f is a real-valued function defined on the real line $\mathbf{R}$ it may happen that equation (2) does not furnish any information whatever about nonzero values of f. This shows that, in general, some further assumptions concerning the greatness of $f^{-1}(\{0\})$ are rather natural. Under such type of assumptions equation (2) has been solved in the class H^G where $(G, +)$ and $(H, +)$ are two abelian groups. The commutativity assumptions were caused by the fact that DE BRUIJN's [3] result on almost additive functions had been used as a tool. With no essential changes Theorem 2 from [8] may be improved by avoiding the commutativity assumptions since the above quoted de Bruijn's result remains valid in the non-abelian case, too (see [7]). Our main purpose here is to extend the de Bruijn's result to the case of almost additive functions defined on semigroups being in a special case embeddable into groups (also without commutativity assumptions). This question is directly connected to the problem of extending of homomorphisms of subsemigroups to homomorphisms of groups (compare [2]). Theorem 1 below yields a joint generalization of de Bruijn's main result from [3] and some theorems from [2]. In section 4 we apply our theorem to Aczél's question regarding equation (1).

§ 2. Let $(G, +)$ be a group (not necessarily commutative) and let $\mathcal{I} \neq 2^G$ be a non-empty family of subsets of G closed under finite unions, hereditary with respect to descending inclusions and such that jointly with a set $U \subset G$ it contains the family $\{x - U\colon x \in G\}$. In the sequel every such family will be called a proper linearly invariant ideal (abbreviated to p.l.i. ideal). The notion of a p.l.i. ideal yields a generalization of null-sets in the theory of Haar-measure and allows to introduce the notion "almost everywhere" in the usual manner (cf. also the section devoted to ideals of negligible sets in [9]). Namely, a property $\mathcal{P}(x)$, $x\in A\subset G$, is said to hold $\mathcal{I}$-almost everywhere in A iff $A\setminus\{x\colon \mathcal{P}(x)\}\in\mathcal{I}$ (for more details and results as well as for examples see [5] and [6]).

Given a p.l.i. ideal $\mathcal{I}$ in $(G, +)$ we put

$$\Omega(\mathcal{I}) := \Big\{M\subset G^2\colon \bigvee_{U(M)\in\mathcal{I}}\ \bigwedge_{x\in G\setminus U(M)} M_x := \{y\in G\colon (x, y)\in M\}\in\mathcal{I}\Big\}.$$

$\Omega(\mathcal{I})$ turns out to be a p.l.i. ideal in $(G^2, +)$.

Given a set $Z\subset G$ we denote by $\mathcal{I}(Z)$ the family of all sets of the form

$$\bigcup_{i=1}^{n}[x_i+Z\cup(-Z)-y_i],$$

where n is a positive integer, $x_i, y_i\in G$, $i=1, \dots, n$, and all their subsets. It is readily seen from this definition that $\mathcal{I}(Z)$ is the smallest set family contained in 2^G such that $Z\in\mathcal{I}(Z)$ and all the conditions occuring in the definition of a p.l.i. ideal except, possibly, that $\mathcal{I}(Z)\neq 2^G$ are satisfied. For this reason $\mathcal{I}(Z)$ is called to be a linearly invariant set ideal generated by Z.

§ 3. Now, suppose that we are given a p.l.i. ideal $\mathcal{I}$ in $(G, +)$ and that $(S, +)$ is a subsemigroup of $(G, +)$ fulfilling the conditions

$$S-S=G \tag{3}$$

and

$$S\notin\mathcal{I}. \tag{4}$$

Remark 1. (3) does not imply (4). Take, for instance, $G=\mathbf{R}^2$ and $S=[0, \infty)^2$. Clearly $S-S=\mathbf{R}^2=G$ whereas the linearly invariant set-ideal $\mathcal{I}(S)$ (generated by S) is proper, i.e. $\mathcal{I}(S)\neq 2^{\mathbf{R}^2}$ *).

We proceed with some lemmas:

Lemma 1. *For every $s, t\in S$ we have $(s+S)\cap(t+S)\notin\mathcal{I}$.*

PROOF. Suppose the contrary, i.e. $(s+S)\cap(t+S)\in\mathcal{I}$ for some $s, t\in S$. Then, for all $u, v\in S$, we have

$$(s+u+S)\cap(t+v+S)\in\mathcal{I}, \tag{5}$$

because of $u+S\subset S$ and $v+S\subset S$. Since $G=S-S=\bigcup_{u\in S}(S-u)$, we may find an $u_0\in S$ such that $v_0:=-s+t+u_0\in S$ whence, in view of (5),

$$t+u_0+S=(s+v_0+S)\cap(t+u_0+S)\in\mathcal{I}.$$

Consequently, $S\in\mathcal{I}$ which contradicts (4).

*) One may also give several further examples; this suprisingly simple one has been suggested to me by M. SABLIK.

Remark 2. A semigroup $(S, +)$ is called to be left reversible iff the intersection $(s+S)\cap(t+S)$ is non-void for any $s, t\in S$ (see e.g. [4]). Thus, Lemma 1 states, in particular, that a semigroup under considerations is left reversible.

Lemma 2. *For every $s, t\in S$ we have $(-s+S)\cap(-t+S)\notin\mathcal{I}$.*

Proof. If we had $(-s+S)\cap(-t+S)\in\mathcal{I}$ for some $s, t\in S$ then, in view of the inclusion $s+S\subset S$, we would also get $S\cap(-t+s+S)\in\mathcal{I}$ and hence $(t+S)\cap(s+S)\in\mathcal{I}$, contrary to Lemma 1.

Lemma 3. *For every sets $U_1, U_2\in\mathcal{I}$ we have*

$$G = (S\setminus U_1)-(S\setminus U_2).$$

Proof. Take an $x\in G=S-S$ and sets $U_1, U_2\in\mathcal{I}$. Then $x=s-t$, $s, t\in S$, and, on account of Lemma 2,

$$T := [-s+(S\setminus U_1)]\cap[-t+(S\setminus U_2)]\notin\mathcal{I}.$$

Thus, $T\neq\emptyset$ and we may find an α such that

$$s+\alpha\in S\setminus U_1 \quad\text{and}\quad t+\alpha\in S\setminus U_2.$$

Now,

$$x = s-t = (s+\alpha)-(t+\alpha)\in(S\setminus U_1)-(S\setminus U_2),$$

which was to be proved.

Lemma 4. *Let $U\in\mathcal{I}$ and $u, s', t'\in S\setminus U$. There exists a pair $(s, t)\in(S\setminus U)^2$ such that $s'-t'=s-t$ and $t\in u+S$.*

Proof. In virtue of Lemma 3 applied to $U_1:=U$ and $U_2:=-u+U$ we get the equality

$$G = (S\setminus U)-[S\setminus(-u+U)]$$

whence

$$G = \bigcup_{x\in S\setminus(-u+U)}[(S\setminus U)-x].$$

This proves that for every $y\in G$ there exists an $x\in S\setminus(-u+U)$ such that $y+x\in S\setminus U$. Take $y:=s'-t'+u$ and a corresponding x. Then

$$s := s'-t'+u+x\in S\setminus U$$

and

$$t := u+x\in(u+S)\setminus U\subset(S\setminus U)\cap(u+S).$$

Evidently, $s-t=s'-t'$ which ends the proof.

Now, assume that we are given two groups $(G, +)$ and $(H, +)$, a p.l.i. ideal $\mathcal{I}$ in $(G, +)$, a subsemigroup $(S, +)$ of $(G, +)$ fulfilling conditions (4) and (5) and a map $f\colon S\to H$ such that

$$f(x+y) = f(x)+f(y) \quad\text{for all}\quad (x, y)\in S^2\setminus M \tag{6}$$

for a certain set $M\in\Omega(\mathcal{I})$. By means of the definition of $\Omega(\mathcal{I})$, there exists a set $U(M)\in\mathcal{I}$ such that $M_x:=\{y\in G\colon (x, y)\in M\}\in\mathcal{I}$ provided $x\in G\setminus U(M)$. We have the following

Lemma 5. *For every* $x, y, u, v \in S \setminus U(M)$ *the equality* $x-y=u-v$ *implies* $f(x)-f(y)=f(u)-f(v)$.

PROOF. Take $x, y, u, v \in S \setminus U(M)$ such that $x-y=u-v$. Lemma 1 ensures that $(y+S) \cap (v+S) \notin \mathcal{I}$. Consequently, $(-v+y+S) \cap S \notin \mathcal{I}$ and hence

$$[-v+y+(S \setminus M_x)] \cap (S \setminus M_u) \notin \mathcal{I}.$$

This enables one to find an

$$s \in ([(-v+y)+(S \setminus M_x)] \cap (S \setminus M_u)) \setminus [M_v \cup (-v+y+M_y)].$$

For such an s we have

$$s \in S, \quad (u, s) \notin M, \quad (v, s) \notin M,$$

$$z := -y+v+s \in S \setminus M_x \quad (\textit{whence } (x, z) \notin M)$$

and

$$(y, z) \notin M.$$

On the other hand

$$x-y+v+s = u-v+v+s = u+s$$

i.e.

$$x+z = u+s$$

whence

$$f(x)+f(z) = f(u)+f(s). \tag{7}$$

The definition of z gives $y+z=v+s$ which implies the equality

$$f(z) = -f(y)+f(v)+f(s).$$

This compared with (7) gives our assertion.

Now, we are able to prove our main result:

Theorem 1. *Let* $(G, +)$ *and* $(H, +)$ *be two groups (not necessarily commutative) and let* $\mathcal{I}$ *be a p.l.i. ideal in* $(G, +)$. *Suppose that* $(S, +)$ *is a subsemigroup of* $(G, +)$ *fulfilling* (3) *and* (4) *and* $f\colon S \to H$ *satisfies the additivity condition* $\Omega(\mathcal{I})$*-almost everywhere in* S^2 *). *Then there exists exactly one additive function* $F\colon G \to H$ *such that* $F|_S = f$ $\mathcal{I}$*-almost everywhere in* S.

PROOF. Suppose that f satisfies (6) for a certain set $M \in \Omega(\mathcal{I})$. Take a $z \in G = [S \setminus U(M)] - [S \setminus U(M)]$ (see Lemma 3). Then $z = x-y$, $x, y \in S \setminus U(M)$. Put

$$F(z) := f(x)-f(y).$$

On account of Lemma 5, the latter formula defines a function $F\colon G \to H$. We shall show that F is additive. For, take $x, y \in G$. We have

$$x = s'-t', \quad y = u-v \quad \text{and} \quad x+y = p-q,$$

with $s', t', u, v, p, q \in S \setminus U(M)$. According to Lemma 4 applied for $U = U(M)$ we may write

$$x = s-t, \quad s, t \in S \setminus U(M)$$

*) In other words, f is $\Omega(\mathcal{I})$-almost additive.

with

$$t \in u+S. \tag{8}$$

Obviously, we have $p-q=x+y=s-t+u-v$, i.e.

$$s-t+u = p-q+v. \tag{9}$$

Since $x+y+q$ and $x+t$ are members of S we infer, by Lemma 1, that $(x+t+S)\cap(x+y+q+S)\notin\mathcal{I}$ whence

$$(-y+t+S)\cap(q+S)\notin\mathcal{I}.$$

Therefore, because of $-y+t=v-u+t$,

$$(v-u+t+S)\cap(q+S)\notin\mathcal{I}$$

and, consequently,

$$(-u+t+S)\cap(-v+q+S)\notin\mathcal{I}.$$

Thus, we may find a

$$w\in[(-u+t+S)\cap(-v+q+S)]\setminus[(-v+q+M_p)\cup(-v+q+M_q)\cup(-u+t+M_s)\cup$$
$$\cup(-u+t+M_t)\cup M_u\cup M_v].$$

For such a w we have

$$w\in -u+t+S\subset S+S\subset S \quad \text{(cf. (8))},$$

$$z_1 := -t+u+w\in S\setminus(M_s\cup M_t),$$

$$z_2 := -q+v+w\in S\setminus(M_p\cup M_q)$$

and

$$(u,w)\notin M,\quad (v,w)\notin M. \tag{10}$$

Relation (9) gives the equality

$$s+z_1 = p+z_2$$

whence, because of $s, p, z_1, z_2\in S$ and $(s,z_1)\notin M$, $(p,z_2)\notin M$, we get

$$f(s)+f(z_1) = f(p)+f(z_2). \tag{11}$$

On the other hand, since

$$t+z_1 = u+w,\quad q+z_2 = v+w,$$

(10) is satisfied as well as $(t,z_1)\notin M$ and $(q,z_2)\notin M$, we may write

$$f(t)+f(z_1) = f(u)+f(w),\quad f(q)+f(z_2) = f(v)+f(w).$$

Therefore, by means of (11),

$$f(s)-f(t)+f(u)+f(w) = f(p)-f(q)+f(v)+f(w)$$

whence

$$[f(s)-f(t)]+[f(u)-f(v)] = f(p)-f(q)$$

i.e.

$$F(x)+F(y) = F(x+y).$$

To prove that $F|_S=f$ $\mathscr{I}$-almost everywhere in S, take an $x\in S\setminus U(M)$. Thus $x=s-t$, $s, t\in S$. Since, in view of Lemma 2, the set

$$(-s+[S\setminus U(M)])\cap(-t+S\setminus[U(M)\cup M_x])$$

is non-void (because it does not belong to $\mathscr{I}$) one may find a $y\in G$ such that

$$s+y\in S\setminus U(M),\quad t+y\in S\setminus(U(M)\cup M_x).$$

Obviously

$$x=s-t=(s+y)-(t+y)$$

which implies

$$F(x)=f(s+y)-f(t+y). \tag{12}$$

On the other hand

$$x+(t+y)=s+y$$

and $(x, t+y)\notin M$. Consequently

$$f(x)+f(t+y)=f(s+y)$$

which compared with (12) gives $F(x)=f(x)$.

To finish the proof it remains to show that F is unique. Suppose that F_1 and F_2 map additively G into H with

$$F_1(s)=F_2(s)=f(s)\quad \text{for}\quad s\in S\setminus U(M)$$

and take an $x\in G$; we have $x=s-t$, $s, t\in S\setminus U(M)$. Thus

$$F_1(x)=F_1(s-t)=F_1(s)-F_1(t)=F_2(s)-F_2(t)=F_2(s-t)=F_2(x),$$

which means that $F_1=F_2$. This completes the proof.

Corollary 1. *Taking $S=G$ we obtain de Bruijn's result* [3] *in the non-abelian case (cf. also* [7]*).*

Corollary 2. *Taking $\mathscr{I}=\{\emptyset\}$ we obtain Theorem* 3 *(and hence also Theorems* 1 *and* 2*) from* [2].

§4. We proceed with the following

Lemma 6. *Let $(S, +)$ be a subsemigroup of a group $(G, +)$ such that $G=S-S$ and let $Z\subset S$ satisfy the condition*

$$\text{(C)}\qquad \begin{cases} \text{for every positive integer } k \text{ and for every } s, s_1, \dots, s_k, \\ t_1, \dots, t_k\in S \text{ there exists a } t\in S+s \text{ such that} \\ t_i+t\notin Z+s_i \text{ and } s_i\notin Z+t_i+t \quad \text{for } i=1, \dots, k. \end{cases}$$

Then the linearly invariant set ideal $\mathscr{I}(Z)$ (generated by Z) does not include S; in particular $\mathscr{I}(Z)$ is proper, i.e. $\mathscr{I}(Z)\neq 2^G$.

Proof (indirect). Suppose that $S\in\mathscr{I}(Z)$, i.e. there exists a positive integer k and elements $x_1, \dots, x_k, y_1, \dots, y_k\in G$ such that

$$S\subset\bigcup_{i=1}^{k}[x_i+Z\cup(-Z)+y_i].$$

Since $G=S-S=\bigcup_{s\in S}(S-s)$, we claim that for every $y\in G$ there exists an $\tilde{s}\in S$ such that $y+\tilde{s}\in S$. Take an $\tilde{s}_1\in S$ such that $y_1+\tilde{s}_1\in S$, an $\tilde{s}_2\in S$ such that $y_2+\tilde{s}_1+\tilde{s}_2\in S$ and so on up to $\tilde{s}_k\in S$ such that $y_k+\tilde{s}_1+\ldots+\tilde{s}_k\in S$. Put $s:=\tilde{s}_1+\ldots+\tilde{s}_k$ and $s_i:=y_i+s$, $i=1, \ldots, k$. Evidently, s and s_i belong to S, $i=1, \ldots, k$. We have

$$S+s\subset\bigcup_{i=1}^{k}[x_i+Z\cup(-Z)+s_i]$$

whence

$$-s-S\subset\bigcup_{i=1}^{k}[-s_i+Z\cup(-Z)-x_i].$$

Repeating the above construction, one can find elements $\tilde{t}$, $t_i\in S$, $i=1, \ldots, k$, such that

$$-s-S+\tilde{t}\subset\bigcup_{i=1}^{k}[-s_i+Z\cup(-Z)+t_i]$$

or, equivalently,

$$-\tilde{t}+S+s\subset\bigcup_{i=1}^{k}[-t_i+Z\cup(-Z)+s_i]$$

whence, in view of the inclusion $S+s\subset-\tilde{t}+S+s$, we get

$$S+s\subset\bigcup_{i=1}^{k}[-t_i+Z\cup(-Z)+s_i].$$

Therefore, for every $t\in S+s$ there exists an $i\in\{1, \ldots, k\}$ such that

$$t\in-t_i+Z+s_i \quad\text{or}\quad t\in-t_i-Z-s_i$$

i.e.

$$t_i+t\in Z+s_i \quad\text{or}\quad s_i\in Z+t_i+t.$$

This contradicts (C) and ends the proof.

Remark 3. Note that in the case where $S\notin\mathscr{I}(Z)$ condition (C) is simply satisfied. In fact, take $s, s_1, \ldots, s_k, t_1, \ldots, t_k\in S$ and

$$t\in(S+s)\setminus\bigcup_{i=1}^{k}[-t_i+Z\cup(-Z)+s_i].$$

Such a t does exist, because $\bigcup_{i=1}^{k}[-t_i+Z\cup(-Z)+s_i]$ belongs to $\mathscr{I}(Z)$ whereas $S+s$ does not. Thus $t\in S+s$ and

$$t_i+t\notin Z+s_i \quad\text{and}\quad s_i\notin Z+t_i+t$$

for $i=1, \ldots, k$. Consequently, condition (C) is equivalent for S not to belong to $\mathscr{I}(Z)$. Observe, however, that (C) involves semigroup terms only.

The assumptions on a semigroup $(S, +)$ we have been doing up to Lemma 6 imply that $(S, +)$ is left reversible (see Remark 2) and cancellative (since $(S, +)$ was a subsemigroup of a given group). It is known (for details, see [4]) that a left reversible semigroup with the cancellation law is embeddable into a group $(G, +)$ in such a manner that $S-S=G$ (*). This together with Lemma 6 enables one to state a theorem on functions $f\colon S\to H$ fulfilling equation (2) with no use of the corresponding group terms.

Theorem 2. *Let $(S, +)$ be a left reversible semigroup (not necessarily commutative) with the cancellation law and let $(H, +)$ be a group (not necessarily commutative). Assume that $f\colon S\to H$ is a solution of (2) such that $Z=f^{-1}(\{0\})$ satisfies condition (C). Then there exists exactly one additive function $F\colon S\to H$ such that $F(x)=f(x)$ for $x\in S\setminus Z$.*

Proof. $(S, +)$ is embeddable into a group $(G, +)$ with $G=S-S$. Moreover, $\mathcal{I}(Z)$ is a p.l.i. ideal in $(G, +)$ and $S\notin\mathcal{I}(Z)$ (see Lemma 6). Consider the set

$$M := \{(x, y)\in S^2\colon x\in Z \text{ or } y\in Z \text{ or } x+y\in Z\}.$$

On account of Lemmas 1 and 2 from [5], $M\in\Omega(\mathcal{I}(Z))$. Clearly, $f(x+y)=f(x)+f(y)$ for $(x, y)\in S^2\setminus M$. Thus f is $\Omega(\mathcal{I}(Z))$-almost additive. Making use of Theorem 1 we infer that there exists exactly one additive function $F\colon S\to H$ such that $E:=\{x\in S\colon f(x)\neq F(x)\}\in\mathcal{I}$. To show that $E\subset Z$ it suffices to repeat the appropriate reasoning applied in the proof of Theorem 2 in [8].

As a corollary we get easily

Theorem 3. *Let $(S, +)$ be a left reversible and cancellative semigroup and let $(K, +, \cdot)$ be a field (both not necessarily commutative). Suppose that a function $g\colon S\to K$ is a solution of (1) such that $Z=g^{-1}(\{0\})$ satisfies condition (C). Then there exists exactly one additive function $F\colon S\to K$ such that $g(x)=\dfrac{1}{F(x)}$ for $x\in S\setminus Z$.*

Finally, we shall present an example in which we are going to visualize that, in our considerations, it was worth-while to handle semigroup terms only (omitting the embedding procedure).

Example. Put $\mathbf{N}:=\{0, 1, 2, \dots\}$, $S:=\mathbf{N}\times 2\mathbf{N}$ and consider a map $+\colon S\times S\to S$ given by the formula

$$(m, x)+(n, y) := (m+n, 2^n x+y), \quad (m, x), (n, y)\in S$$

(the sign $+$ on the right hand side denotes the usual addition in $\mathbf{N}$). It is not hard to check that the pair $(S, +)$ yields a cancellative and left reversible semigroup(**) with $(0, 0)$ as a neutral element. In spite of the fact that the set

$$Z_0 := \{(p, z)\in S\colon z\leqq p\}$$

is rather "large" in S, we are able to determine all the solutions $g\colon S\to K$ (with $(K, +, \cdot)$ — an arbitrary field) of equation (1) which satisfy the condition $g^{-1}(\{0\})\subset Z_0$. For, we shall show that Z_0 satisfies condition (C). In fact, take $s=(n, y)$, $s_i=(n_i, y_i)\in S$ and $t_i=(m_i, x_i)\in S$, $i=1, \dots, k)$. We have to find a pair

$$(m, x) = t\in S+s = \{(p+n,\ 2^n z+y)\in S\colon (p, z)\in S\}$$

(*) Obviously, every commutative semigroup is left reversible. Therefore, every commutative semigroup $(S, +)$ is embeddable into a group $(G, +)$ (with $G=S-S$) if and only if it is cancellative.

(**) It is not right reversible (compare [4], Exercise 1 for § 12.4).

such that

(13) $(m_i+m, 2^m x_i+x) = t_i+t \notin Z_0+s_i = \{(p+n_i, 2^{n_i}z+y_i)\in S: z \leqq p\}$

and

(14) $(n_i, y_i) = s_i \notin Z_0+t_i+t = \{(p+m_i+m, 2^{m_i+m}z+2^m x_i+x)\in S: z \leqq p\}$

for $i=1, \dots, k$. In order to have $t=(m, x)\in S+s$, take $m=n$ and $x=2^n z_x+y$ with $z_x\in 2\mathbf{N}$(unrestricted temporarily). To realize (13) and (14) (with $m=n$) for $i=1, \dots, k$, denote by P the set of all $p\in\mathbf{N}$ such that p is a solution of at least one of the equations

$$m_i+n = p+n_i, \quad n_i = p+m_i+n, \quad i = 1, \dots, k,$$

and put $p_0:=\max P$. Obviously, for all $p>p_0$, $p\in\mathbf{N}$, and all $z\in 2\mathbf{N}$ we have

(13′) $$(m_i+n, 2^n x_i+x) \neq (p+n_i, 2^{n_i}z+y_i)$$

and

(14′) $$(n_i, y_i) \neq (p+m_i+n, 2^{m_i+m}z+2^n x_i+x)$$

for $i=1, \dots, k$ (independently of the choice of x). If one has $p\leqq p_0$ and $(p, z)\in Z_0$ then necessarily $z\leqq p_0$ whence, in order to get (13′) and (14′) for $i=1, \dots, k$, it suffices to take x large enough (which may be done by making z_x large enough).

Consequently, Z_0 satisfies condition (C) (and, obviously, so does an arbitrary subset of Z_0). According to Theorem 3, a function $g: S\to K$ fulfilling (1) and the condition $Z:=g^{-1}(\{0\})\subset Z_0$ is of the form

$$g(x) = \begin{cases} 0 & \text{for} \quad x\in Z \\ \dfrac{1}{F(x)} & \text{for} \quad x\in S\setminus Z \end{cases}$$

where F is an arbitrary homomorphism of S into K. Now, we have to find a representation of such homomorphisms, i.e. to solve the functional equation

(15) $$F(m+n, 2^n x+y) = F(m, x)+F(n, y), \quad (m, x),\ (n, y)\in S.$$

Putting $\varphi(m):=F(m, 0)$, $m\in\mathbf{N}$, $\psi(y):=F(0, y)$, $y\in 2\mathbf{N}$, and setting $n=x=0$ in (15) we get

$$F(m, y) = \varphi(m)+\psi(y), \quad (m, y)\in S.$$

Setting $x=y=0$ and, subsequently, $m=n=0$ in (15) we obtain the relations

$$\varphi(m+n) = \varphi(m)+\varphi(n), \quad (m, n)\in \mathbf{N}^2,$$

$$\psi(x+y) = \psi(x)+\psi(y), \quad (x, y)\in (2\mathbf{N})^2,$$

which imply easily

$$\varphi(m) = m\alpha, \quad m\in\mathbf{N}, \quad \psi(y) = y\beta, \quad y\in 2\mathbf{N},$$

where α, β are certain constants from K. Thus $F(m, y)=m\alpha+y\beta$, $(m, y)\in S$, which inserted to (15) gives $\beta=0$. Therefore

$$g(m, x) = \begin{cases} 0 & \text{for} \quad (m, x)\in Z \\ \dfrac{1}{m\alpha} & \text{for} \quad (m, x)\in S\setminus Z. \end{cases}$$

References

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