Note on maximal asymptotic nonbases of zero density

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M. B. NATHANSON introduced in [2] the notion of maximal asymptotic nonbasis as the dual of the notion of minimal asymptotic basis. We call a strictly increasing sequence of nonnegative integers a maximal asymptotic nonbasis of order h, if it possesses the following two properties;

(i) A is not an asymptotic basis of order h,

(ii) if b is any nonnegative integer and $b \notin A$, then $A \cup b$ is an asymptotic basis of order h.

In the above mentioned paper Nathanson showed that under certain conditions the union of suitable residue classes satisfies (i) and (ii) and yields therefore a maximal asymptotic nonbasis of positive density. In [2] Nathanson posed the question of the existence of a maximal asymptotic nonbasis of order $h \geqq 2$, for which $\liminf_x \frac{A(x)}{x} = 0$ (as usual, $A(x)$ denotes the number of the elements of the sequence A which are not greater than x).*)

In our paper [4] we gave an affirmative answer to this open question by Nathanson in the case $h=2$. In the present paper we continue these investigations and we show that there exist such maximal second order nonbasis A of zero density, for which $A(x)=O(\sqrt{x})$ and this estimate is already best possible. Furthermore we show that from any second order basis $A=\{a_1, a_2, \dots\}$ satisfying $\sup_{i=1,2,\dots} (a_{i+1}-a_i)=\infty$ a maximal second order asymptotic nonbasis A^* can be constructed for which $A^*(x) \leqq 12A(x)$ holds. As is known, in*) [1] and [3] one can find examples for second order basis with the property $A(x)=O(\sqrt{x})$.

Definition. The sequence A^* is called a transformed of the sequence A_0 with respect to the sequences $\{m_i\}$, $\{M_i\}$, $\{b_i\}$, if $x \leqq m_k$ and $x \in A_k$ imply $x \notin A^*$ and conversely $(k=1, 2, \dots)$, where

$$A_k = A^{\mathrm{I}}_{k-1} \cup A^{\mathrm{II}}_{k-1} \cup A^{\mathrm{III}}_{k-1} \cup A^{\mathrm{IV}}_{k-1} \quad (k=1, 2, \dots),$$

and

$$A^{\mathrm{I}}_{k-1} = \{x \mid x \leqq m_k \text{ and } x \in A_{k-1}\},$$

$$A^{\mathrm{II}}_{k-1} = \{x \mid x = M_k - y,\ 0 \leqq y \leqq b_k \text{ and } y \notin A_{k-1}\},$$

$$A^{\mathrm{III}}_{k-1} = \{x \mid x = M_k + a_i + 1,\ a_i \in A_{k-1}\},$$

$$A^{\mathrm{IV}}_{k-1} = \{2M_k+1,\ M_k+M_{k-1}+1,\ M_k+M_{k-2}+1,\ \dots,\ M_k+M_1+1\}.$$

*) Added in proof: Recently M. B. NATHANSON (*J. London Math. Soc.* (2) **15**, 1977, 29—34) has proved the existence of a maximal second order asymptotic nonbasis with $A(x)=O(\sqrt{x})$

The sequences A_0 and A^* are interesting for us, in general, if their elements are nonnegative integers. This is why we assume that the sets $A^{I}_{k-1}, A^{II}_{k-1}, A^{III}_{k-1}, A^{IV}_{k-1}$ do not possess negative elements. If for instance on grounds of the choice of the sequence $\{M_i\}$ the set A^{IV}_{k-1} had a negative number as element, then we consider the set A^{IV}_{k-1} empty.

Note that with the choice $\{m_i \equiv 1\}$, $\{M_i \equiv -1\}$ and $\{b_i \equiv 0\}$ the transformed of an arbitrary sequence A_0 (which consists of nonnegative elements) consides with A_0. By definition the set A^{II}_0 is then empty, and it follows that $A^{I}_0 \subset A_0$ and $A^{III}_0 \equiv A$; then by our assumption A^{IV}_0 is likewise to be considered empty. This means that a suitable transformed of a maximal asimptotic nonbasis of second order is very same.

In the sequel we consider the question, when a transformed of a second order basis is a second order maximal nonbasis. Our purpose is to find a sufficient condition.

The transformed of a second order basis A_0 can be a second order maximal nonbasis, if to an arbitrary positive integer b there exists an $a_i \in A_0$ such that for the representation of numbers of (a_i, a_i+b) (as the sum of two elements of A_0) a number of (a_i, a_i+b) is not necessary. If this latter condition is not fulfilled, then $(a_{i+1}-a_i)$ must be smaller than b, than is

$$\max_i (a_{i+1}-a_i) \leqq b, \quad a_{i+1}, a_i \in A_0.$$

The sequence $\{0, 1, 2, 3, 6, \dots, 3k, \dots\}$ is for $b=7$ an example for the above mentioned property. From the inequality $\max_i (a_{i+1}-a_i) \leqq b$, $a_i, a_{i+1} \in A_0$ it does not yet follow that there exists to the sequence A_0 a number b such that there is no a_i for which a number from (a_i, a_i+b) is necessary in order to represent the elements of (a_i, a_i+b). An example for such a sequence is given if the numbers $\{2^2, 2^3, \dots, 2^k, \dots\}$ are omitted from the nonnegative integers.

A series A_0 can evidently possess many transformed, and a transformed may belong to more than one sequence.

Denote by $A(x)$ the number of those elements of the sequence A which are not greater than x.

Theorem. *If a second order basis A_0 satisfies the condition* $\sup_{i=1,2,\dots} (a_{i+1}-a_i) = \infty$, *then it has a transformed A^* such that A^* is a second order maximal asymptotic nonbasis and $A^*(x) \leqq 12 A_0(x)$.*

Proof. Our main point in the proof is a good choice of the sequences

$$\{m_1, \dots, m_k, \dots\}, \quad \{M_1, \dots, M_k, \dots\} \quad \text{and} \quad \{b_1, \dots, b_k, \dots\},$$

If we succeed in properly defining these sequences, then it will be easier to show the indicated property of A^*. This is why more place is devoted to the construction of the sequences $\{m\}$, $\{M\}$ and $\{b\}$ than to the investigation of the corresponding properties of the sequence A^*.

Let $2 \leqq b_1$ be an integer.

Denote by ${}_1a_i$ the smallest element of A_0 for which the following two conditions are satisfied:

$$(1) \qquad {}_1a_i + b_1 < {}_1a_{i+1}$$

and

$$(2) \qquad b_1 < \frac{1}{3} \log A_0({}_1a_i)$$

where ${}_1a_{i+1}$ is that element of the sequence A_0 which follows ${}_1a_i$, and we choose $m_1 = {}_1a_i$ for m_1. Because of the condition $\sup_{i=1,2,\dots} (a_{i+1} - a_i) = \infty$ it is evident that such $a_i = {}_1a_i$ exist. We define M_1 as the smallest integer to the representation of which as a sum of two elements of A_0 one needs an element of A_0 that is greater than ${}_1a_i$.
Then

$$M_1 > {}_1a_i + b_1$$

for ${}_1a_i$ has been chosen in such a way that A does not contain elements in the interval $({}_1a_i, {}_1a_i + b)$. Starting with A_0 we show that by definition one gets a sequence A_1 for which

(3) $A_1(x) = A_0(x) + O(M_1)$ for any sufficiently large x,

(4) $A_1(x) \leqq 3A_0(x)$ for every x,

(5) $M_1 \notin 2A_1$ but if $x \neq M_1$ and $x \geqq 0$ integer, then $x \in 2A_1$,

(6) if $x \notin A_1$ and $0 \leqq x \leqq b_1$, then $M_1 \in 2\{A_1 \cup \{x\}\}$.

To shows (3) and (4) it will be sufficient to investigate $A_1(x)$ on three intervals.

If $0 \leqq x \leqq M_1 - b_1$, then $A_1(x) \leqq A_0(x)$ because in case $x \leqq m_1$, $A_1(x) = A(x)$ and we ommitted according to the definition the elements of A_0 which are between m_1 and $M_1 - b_1$; for A_0^{I} has elements smaller than m_1 and $A_0^{\mathrm{II}}, A_0^{\mathrm{III}}, A_0^{\mathrm{IV}}$ have elements greater than $M_1 - b_1$. If $M_1 - b_1 \leqq x \leqq M_1$, then let

$$A_1(x) = A_1(M_1 - b_1) + r(x),$$

where $A_1(M_1 - b_1) \leqq A_0(M_1 - b_1) \leqq A_0(x)$ and $r(x) = b_1$, since we can take at most b_1 elements to A_0 from the interval $(M_1 - b_1, M_1)$.

The relation $r(x) \leqq b_1 < \frac{1}{3} \log A_0({}_1a_i) < \frac{1}{3} \log A_0(M_1)$ follows from condition (2) and from the fact that M_1 is evidently greater than ${}_1a_i$.

If $M_1 < x$, then

$$A_1(x) \leqq A_1(M_1 - b_1) + r(M_1) + A_0\big(x - (M_1 + 1)\big) + 1$$

since the number of elements of A_0^{I} and A_0^{II} is exactly $A_1(M_1 - b_1) + r(M_1)$, on the other hand, the elements which are greater than M_1 belong to A_0^{III} and of these the number of the elements not greater than x is equal to

$$A_0(x - M_1 - 1).$$

The inequality

$$A_1(M_1 - b_1) + r(M_1) + A_0(x - M_1 - 1) + 1 \leqq$$

$$\leqq A_0(M_1) + \frac{1}{3} \log A_0(M_1) + A_0(x) + 1$$

holds because of the inequalities $r(M_1) \leqq b_1 < \frac{1}{3} \log A_0(M_1)$, $A_1(M_1 - b_1) = A_0(M_1)$

and $A_0(x-M_1-1) \leqq A_0(x)$. Therefore we can write

$$A_1(x) = A_0(x)+O(M_1) \tag{7}$$

and

$$A_1(x) \leqq 3A_0(x). \tag{8}$$

In order to prove (5) we consider the following. $2A_1$ contains the numbers smaller than M_1. If $M_1+1 \leqq x \leqq 2M_1+1$, then x can be represented in the form $x=a_j+M_1+a_i+1$, where $a_j, M_1+a_i+1 \in A_1$, and $x=2M_1+1$ is represented in the form $x=2M_1+1+0$ with $0, 2M_1+1 \in A_1$. If $x \geqq 2M_1+2$, then x can be represented as $x=M_1+a_r+1+M_1+a_s+1$, where M_1+a_r+1, $M_1+a_s+1 \in A_1$. Thus (5) is verified. One obtains (6) immediately because of the definition of transforming; $x<b_1$ and $x \notin A_1$, $x \geqq 0$ yield $x \notin A_0$ and then $M_1-x \in A_0^{\mathrm{II}}$, that is, $M_1-x \in A_1$ and thus $M_1-x+x=M_1 \in 2\{A_1 \cup \{x\}\}$. For A_1 the condition $\sup\limits_{i=1,2,\dots} (a_{i+1}-a_i)=\infty$ is satisfied, since A_1 has been obtained from A_0 by changing finitely many elements and adding to every element a fixed number.

Let now $k \geqq 2$ be an arbitrary natural number, $b_k=b_{k-1}+1$ and let $c_1, c_2, c_3, \dots$ be real numbers greater than 1 for which

$$\prod_{i=1}^{\infty} c_i = 4 \tag{$*$}$$

holds. Assume that $A_1, \dots, A_{k-1}$ have already been constructed with the desired properties corresponding to (3)—(6) and to the conditions of the theorem.

We choose an ${}_k a_i$ from A_{k-1} in such a way that the following properties are satisfied:

$$\text{if} \quad x \geqq {}_k a_i, \quad \text{then} \quad \frac{A_{k-1}(x)}{A_{k-2}(x)} < c_{k-1},$$

$${}_k a_i + b_k < {}_k a_{i+1}, \quad b_k < \frac{1}{3} \log\left(A_{k-1}({}_k a_i)\right),$$

and

$$m_k = {}_k a_i.$$

Let $b_k=b_{k-1}+1$ and let M_k be the smallest natural number which cannot be represented as the sum of two elements of A_{k-1} not exceeding ${}_k a_i$. From A_{k-1} we obtain A_k according to the definition, with the given values m_k, M_k, b_k. It remains yet to show that A_k has the following properties:

$$A_k(x) = A_{k-1}(x)+O(M_k) \tag{9}$$

for any sufficiently large x;

$$A_k(x) \leqq 3A_{k-1}(x) \quad \text{for every } x; \tag{10}$$

$$M_1, M_2, \dots, M_k \notin 2A_k, \text{ but if } x \neq M_i \ (i=1, \dots, k) \tag{11}$$

and $x \geqq 0$ integer, then $x \in 2A_k$;

$$\text{if} \quad x \notin A_k \quad \text{and} \quad 0 \leqq x < b_k, \quad \text{then} \quad M_k \in 2\{A_k \cup \{x\}\}. \tag{12}$$

The proof of (9), (10) is the same as that of (3), (4); we only have to take into consideration the changes

$$A_0 \to A_{k-1}, \quad A_1 \to A_k, \quad b_1 \to b_k, \quad M_1 \to M_k.$$

Note that in case $M_k < x$,

$$A_k(x) \leqq A_{k-1}(M_k) + b_k + k + A_{k-1}(x - M_k - 1),$$

but $k < b_k$, and thus again the inequality

$$A_k(x) \leqq 3A_{k-1}(x)$$

holds for every x.

For the proof of (11) we note that $M_1, M_2, \dots, M_k \notin 2A_k$ because M_i $(i = 1, 2, \dots, k)$ can be represented at most by a sum of elements of A^{I}_{i-1} and A^{II}_{i-1}; this, however, is not possible by definition of A^{I}_{i-1} and A^{II}_{i-1}.

Now we prove (11), that is, we show that $x \neq M_i$ $(i = 1, \dots, k)$ implies $x \in 2A_k$.

If $x < M_k$, then we are done because x is contained in $\{A^{\mathrm{I}}_{k-1} + A^{\mathrm{I}}_{k-1}\} \subset 2A_k$. If x is in the interval $[M_k + 1, 2(M_k + 1)]$ then $x - M_k - 1 = D \neq M_i$ $(i = 1, \dots, k)$ implies $D = a_r + a_s$, $a_r + a_s \in 2A^{\mathrm{I}}_{k-1}$, that is, because of $x = a_r + a_s + M_k + 1$ and $a_r \in A^{\mathrm{I}}_{k-1}$, $a_s + M_k + 1 \in A^{\mathrm{III}}_{k-1}$ we obtain $x \in A^{\mathrm{I}}_{k-1} + A^{\mathrm{III}}_{k-1}$ and thus $x \in 2A_k$. If $x = M_k + 1 + M_i$ $(i = 1, 2, \dots, k)$, then $0 \in A^{\mathrm{I}}_{k-1}$ and $M_k + 1 + M_i \in A_k$ imply $x \in 2A_k$. If $x \geqq 2M_k + 2$ and $x - (2M_k + 2) = M \neq M_i$, then $M = a_i + a_j$, where $a_i, a_j \in A_{k-1}$ and thus $a_i + M_k + 1$, $a_j + M_k + 1 \in A^{\mathrm{III}}_{k-1}$ that is, $x \in 2A_k$. If however $M = M_i$ then we take into consideration that $M_i + M_k + 1$ and $M_k + 1$ belong to A_k and obtain $x \in 2A_k$. To show (12) it is sufficient to note that $M_k - x \in A^{\mathrm{II}}_{k-1}$ because of the conditions $x \notin A_{k-1}$ and $0 \leqq x < b$. In this case $x + (M_k - x)$ is indeed an element of $A_k + \{x\}$. For A_k the condition $\sup_{a_i \in A_k} (a_{i+1} - a_i) = \infty$ is satisfied, since it has been constructed from A_{k-1} by changing a finite number of elements or by shifting of A_{k-1} with a fixed number.

By the previous recursive definition of the sequences $\{m\}$, $\{M\}$ and $\{b\}$ we constructed step by step the sequence A^*, too. For, if we transform with (m_i, M_i, b_i) then those elements of the sequence which are smaller than m_i remain unchanged. Properties (5) and (11) show that $\{M\} \cap 2A^* = \emptyset$ and that $x \notin \{M\}$ implies $x \in 2A^*$. From (6) and (12) it follows that for an arbitrary $x > 0$ and $x \notin A^*$ there exists a k such that $x < b_k$. Then $i > k - 1$ implies $M_i - x \in A''_{i-1}$, and thus $\in A^*$. This means that A^* is in fact a second order maximal asymptotic nonbasis.

We have yet to show that $\dfrac{A^*(x)}{A_0(x)} \leqq 12$. Let x be an arbitrary positive real number. Then for some k one has $m_{k-1} < x < m_k - b_k$, that is,

$$A^*(x) = A_k(x) = A_{k-1}(x) \quad \text{and} \quad \frac{A_{k-1}(x)}{A_{k-2}(x)} < 3.$$

Thus $A_{k-1}(x) < 3A_{k-2}(x)$. On the other hand, $3A_{k-2}(x) < 3c_{k-2}A_{k-3}(x)$ since ${}_{k-1}a_i$ has been chosen in such a way that $x \geqq {}_{k-1}a_i$ implies

$$\frac{A_{k-2}(x)}{A_{k-3}(x)} < c_{k-2}.$$

The inequality $x \geqq {}_{k-1}a_i$ holds on account of $x > m_{k-1}$. Using

$$x \geqq {}_{k-1}a_i \geqq {}_{k-2}a_i \geqq \dots \geqq {}_1a_i,$$

it follows that

$$3A_{k-2}(x) < 3c_{k-2}A_{k-3}(x) < 3c_{k-2}c_{k-3}A_{k-4}(x) < \dots < 3c_{k-2}\dots c_1 A_0(x),$$

that is, $A^*(x) < 3c_{k-2}\dots c_1 A_0(x)$. From this we obtain by the choice of $c_1, c_2, \dots$ the relation $c_{k-2}\dots c_1 < 4$ and therefore

$$A^*(x) < 12A_0(x).$$

Remark. The constant 12 occurring in the theorem can be improved to $2+\varepsilon$, where $\varepsilon > 0$ is an arbitrarily small constant.

First we show that in (10) we can write $2+\varepsilon_1$ instead of 3, where $\varepsilon_1 > 0$ is an arbitrarily small number. $A_k(x) \leqq A_{k-1}(M_k) + b_k + A_{k-1}(x - M_{k-1}) + k$ holds for every $x > M_k$, but because of $A_{k-1}(x - M_{k-1}) \leqq A_{k-1}(x)$ and $b_k > k$ one has $A_k(x) = A_{k-1}(M_k) + A_{k-1}(x) + 2b_k$. The M_k and the b_k must be chosen in such a way that $\varepsilon_1 A_{k-1}(M_k) > 2b_k$ which can be done without difficulty. Taking into consideration that $A_k(x) - A_{k-1}(x) \leqq$ constant for $x > M_k$, we get

$$A_k(x) \leqq (2+\varepsilon_1)\, A_{k-1}(x)$$

for any x.

If in $(*)$ the constants c_i are chosen in such a way that $\prod_{i=1}^{\infty} c_i = 1+\varepsilon_2$ and $c_i > 1$ for every i, then we obtain

$$A^*(x) < (2+\varepsilon_1)(1+\varepsilon_2)\, A(x)$$

instead of $A^*(x) \leqq 12A_0(x)$. If ε_1 and ε_2 are sufficiently small positive numbers, then

$$A^*(x) < (2+\varepsilon)\, A(x).$$

It is not difficult to provide an example for a sequence $c_1, c_2, \dots$ satisfying the conditions $\prod_{i=1}^{\infty} c_i = 1+\varepsilon_2$ and $c_i > 1$ for $i = 1, 2, \dots$ with arbitrary $\varepsilon_2 > 0$. Choose for instance $\alpha > 1$ such that $\alpha^\beta = 1+\varepsilon_2$ and let β be an irrational number. Such numbers α and β evidently exist. Write β as a decimal fraction, that is, $\beta = \sum_{i=l}^{-\infty} \gamma_i \cdot 10^i$. Let then $c_k = \alpha^{\gamma_{i_k} 10^{i_k}}$ where γ_{i_k} is the k-th figure of β differing from 0. Then obviously $\prod_{k=1}^{\infty} c_k = 1+\varepsilon_2$.

Corollary 1. *There exists a second order maximal asymptotic nonbasis A^* such that for any positive x the condition $c_1\sqrt{x} \leqq A^*(x) \leqq c_2\sqrt{x}$ is satisfied with suitable positive constants c_1, c_2.*

Proof. In [1] and [3] is proved that there exists a second order basis A_0 such that $A_0(x) \leqq c\sqrt{x}$ with a suitable constant c. If we apply our theorem to such a sequence A_0 (this can be done since A_0 has the density 0, that is, the condition

$\sup_{i=1,2,\ldots} (a_{i+1}-a_i)=\infty$ is obviously satisfied for A_0), then A_0 has a transformed A^* having the property that $A^*(x)\leqq 12A_0(x)\leqq c_2\sqrt{x}$. Furthermore there exists a natural number which can be taken to A^* in order to obtain a second order basis. Thus according to [3] there are at least $c_1\sqrt{x}$ elements of $A^*(x)$ not exceeding x where c_1 denotes a suitable positive constant.

Corollary 2. *If A_0 is a second order basis of zero density, then there exists a transformed A^* of A which is also of zero density.*

Proof. It suffices to show that the condition $\sup_{i=1,2,\ldots} (a_{i+1}-a_i)=\infty$ is satisfied for sequences of zero density. Then according to our theorem the sequence A_0 has a transformed A^* for which $A^*(x)\leqq 12A_0(x)$ and A^* is a second order maximal asymptotic nonbasis. Furthermore then

$$\lim_{x\to\infty} \frac{A^*(x)}{x} \leqq \lim_{x\to\infty} \frac{12A_0(x)}{x} = 12 \lim_{x\to\infty} \frac{A_0(x)}{x} = 0,$$

that is, A^* is in fact a sequence of zero density. The condition $\sup_{i=1,2,\ldots} (a_{i+1}-a_i)=\infty$ is satisfied for sequences of zero density because of the following lemma (which asserts more than we shall here need).

Lemma. (cf. [4]). *If the sequence A satisfies the condition $\lim_{x\to\infty} \frac{A(x)}{x}=0$, then for an arbitrarily small $\varepsilon>0$ and for an arbitrarily great number M there exist $a_i\in A$ such that the following conditions hold:*

$$\frac{A(a_i)}{a_i}<\varepsilon, \tag{13}$$

$$a_i-a_{i-1}>M, \tag{14}$$

$$\text{if} \quad i>j, \quad \text{then} \quad a_i-a_{i-1}>a_j-a_{j-1}. \tag{15}$$

Proof. This is Lemma 1. in [4].

In general, it is not true that to an A_0 satisfying the assumption of the theorem there exists an constant c_3 such that for any x and for any A^* the relation $A_0(x)\leqq \leqq c_3A^*(x)$ holds. If we take a second order basis B such $B(x)=O(\sqrt{x})$, than from this one can derive a sequence which may serve as a counterexample. Consider for instance a sequence $M=\{M_1, M_2, \ldots\}$ for which $M_i\geqq M_{i-1}^{M_{i-1}}$ ($i=2, 3, \ldots$). Let $B'=H\cup B$ where $H=\bigcup_{i=1}^{\infty}\{M_i, M_i^2\}$ and where $\{M_i, M_i^2\}$ denotes the set of those integers Y for which $M_i<Y<M_i^2$. Let us now investigate the quotient $\frac{B_1'(x)}{B'(x)}$. Then for $x=M_1$ we approximately obtain $\frac{1}{\sqrt{M_1}}$, whereas for $x=M_1^2$ the quotient is approximately 1. Thus a good estimation is not possible.

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