On a problem of Evelyn—Linfoot and Page in additive number theory

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§ 1. *Introduction.* Let $k\geqq 2$ be a fixed integer. A positive integer is called k-free if it is not divisible by the k-th power of any prime. Let Q_k denote the set of k-free integers. For each positive integer n, let $T_k(n)$ denote the number of pairs $(a, b)\in Q_k\times Q_k$ with $n=a+b$. In 1931, EVELYN and LINFOOT [5] established the following asymptotic formula for sufficiently large n and every $\varepsilon>0$:

$$T_k(n) = c_k \varrho_k(n)n+E_k(n), \tag{1.1}$$

where

$$E_k(n) = O(n^{2/(k+1)+\varepsilon}), \tag{1.2}$$

$$c_k = \prod_p (1-2p^{-k}),$$

$$\varrho_k(n) = \prod_{p^k|n} (1+(p^k-2)^{-1}),$$

where p runs over all primes. Another proof was given by ESTERMANN [3] in the same year.

In 1932, PAGE [8] generalized the Evelyn—Linfoot formula as follows: let k, l be fixed integers with $2\leqq k\leqq l$ and let $T_{k,l}(n)$ denote the number of pairs $(a, b)\in Q_k\times Q_l$ with $n=a+b$. Then for sufficiently large n and every $\varepsilon>0$:

$$T_{k,l}(n) = c_{k,l}\varrho_{k,l}(n)n+E_{k,l}(n), \tag{1.3}$$

where

$$E_{k,l}(n) = O(n^{(k+l-2)/(kl-1)+\varepsilon}), \tag{1.4}$$

$$c_{k,l} = \prod_p (1-p^{-k}-p^{-l}),$$

$$\varrho_{k,l}(n) = \prod_{p^k|n} (1+(p^l-p^{l-k}-1)^{-1}).$$

It is clear that (1.3) reduces to (1.1) in the case $k=l$.

Denote by $Q_k(x; a, H)$ the number of k-free numbers $n\leqq x$ with $n\equiv a \pmod H$. In 1960, SUBHANKULOV and MUHTAROV [10] gave a new "proof" of (1.1), (1.2) in the case $k=2$. Their idea was to use an estimate for $Q_2(x; a, H)$. However, there is a serious error in their work (the estimate in the third line from the bottom of p. 4 is not valid for large d) which invalidates the lemma of their paper [10]. In 1963, COHEN and ROBINSON [2] showed how an estimate of $Q_k(x; a, H)$ could be used to prove (1.3), but without an explicit error term. In 1965, COHEN [1] using an estimate for $Q_2(x; a, H)$ particularly accurate when x is not much bigger than H, gave a new proof of (1.1), (1.2) in the case $k=2$, achieving the slightly better error estimate

$$E_2(n) = O(n^{2/3} \log^2 n). \tag{1.5}$$

In this paper we give a new proof of the Page formula (1.3), (1.4) using an estimate of $Q_k(x; a, H)$ due to PRACHAR [9] that is also particularly accurate for small x. Perhaps surprisingly our new proof gives a considerable sharper error estimate in the case $k \neq l$. In fact we improve (1.4) to

$$E_{k,l}(n) = \begin{cases} O(n^{(k+l)/l(k+1)+\varepsilon}, & \text{if} \quad l \leqq k^2 \\ O(n^{1/k}), & \text{if} \quad l > k^2 \end{cases} \tag{1.6}$$

for every $\varepsilon > 0$. Paying more attention to small errors in the case $l \leqq k^2$ we can improve on (1.6) getting

$$E_{k,l}(n) = O\big(n^{(k+l)/l(k+1)} (\log n)(k^{1/(k+1)} - 1)(l-1)/l\big). \tag{1.7}$$

Note that in the case $k=l=2$, (1.7) implies $E_{2,2^{(n)}} = E_2(n) = o(n^{2/3} \log^{13/100} n)$ which is an improvement over (1.5).

We denote by $\omega(n)$ the number of distinct prime factors of n, $\tau(n)$ the number of positive divisiors of n, μ the Möbius function, ζ the Riemann ζ-function.

§ 2. *Proof of (1.6).* We begin with a statement of the result of Prachar [9] referred to above. We call an integer n k-full, if $p^k | n$ for every prime $p|n$.

Theorem 2.1. *If $(a, H) \in Q_k$ and H is k-full, then*

$$Q_k(x; a, H) = A_{k,H} \frac{x}{H} + O\big(k^{\omega(H)} (x^{1/k} H^{-1/k^2} + H^{1/k})\big) \tag{2.1}$$

where

$$A_{k,H} = \sum_{\substack{d=1 \\ (d,H)=1}}^{\infty} \frac{\mu(d)}{d^k} = \frac{1}{\zeta(k) \prod_{p|H} (1-p^{-k})}. \tag{2.2}$$

The implied constant in the O-estimate is absolute.

Remark. Prachar proves (2.1) under the assumption $(a, H)=1$, rather than the assumption $(a, H) \in Q_k$ and H is k-full. However the proof in either case is almost identical (cf. § 3). Note that if $(a, H) \notin Q_k$, then $Q_k(x; a, H)=0$ for all x.

Theorem 2.2. *For sufficiently large integers n*

$$E_{k,l}(n) = \begin{cases} O(n^{(k+l)/l(k+1)+\varepsilon}), & \text{if} \quad l \leqq k^2 \\ O(n^{1/k}), & \text{if} \quad l > k^2. \end{cases}$$

PROOF. Let $Q_k^*(x; a, H)$ denote the number of $n < x$ with $n \in Q_k$, $n \equiv a \pmod{H}$. Then $Q_k^*(x; a, H)$ differs from $Q_k(x; a, H)$ by at most 1. Let $t = t(n)$ be arbitrary with $1 < t < n^{1/l}$. From the identity ([2], p. 291)

$$T_{k,l}(n) = \sum_{h < n^{1/l}} \mu(h) Q_k^*(n; n, h^l),$$

we have

$$T_{k,l}(n)=\sum_{h\leq t}\mu(h)Q_k(n;n,h^l)+\sum_{t<h<n^{1/l}}\mu(h)Q_k(n;n,h^l)+O(n^{1/l})= \tag{2.3}$$
$$=T_1+T_2+O(n^{1/l}),\quad \text{say.}$$

Since $Q_k(n;n,h^l)<2nh^{-l}$ for $h<n^{1/l}$, we have

$$|T_2|\leq\sum_{t<h}2nh^{-l}=O(nt^{1-l}). \tag{2.4}$$

By Theorem 2.1 we have

$$T_1=\sum_{\substack{h\leq t\\(n,h^l)\in Q_k}}\mu(h)\left\{A_{k,h^l}\frac{n}{h^l}+O\left(k^{\omega(h)}(n^{1/k}h^{-l/k^2}+h^{l/k})\right)\right\}=T_{11}+T_{12}+T_{13},\quad \text{say.}$$

Now using (2.2) we have

$$T_{11}=\frac{n}{\zeta(k)}\sum_{\substack{h\leq t\\(n,h^l)\in Q_k}}\mu(h)h^{-l}\prod_{p|h}(1-p^{-k})^{-1}= \tag{2.5}$$
$$=\frac{n}{\zeta(k)}\sum_{\substack{h=1\\(n,h^l)\in Q_k}}^{\infty}\mu(h)h^{-l}\prod_{p|h}(1-p^{-k})^{-1}+O\left(n\sum_{h>t}h^{-l}\right)=c_{k,l}\varrho_{k,l}(n)n+O(nt^{1-l})$$

using the infinite series evaluation of [2], p. 292. Also

$$T_{12}=O\left(n^{1/k}\sum_{h\leq t}k^{\omega(h)}h^{-l/k^2}\right)=O\left(n^{1/k}\sum_{h\leq t}h^{-l/k^2+\varepsilon}\right)=\begin{cases}O(n^{1/k}t^{1-l/k^2+\varepsilon}), & \text{if } l\leq k^2\\ O(n^{1/k}), & \text{if } l>k^2,\end{cases}$$

where we use the fact that $k^{\omega(h)}=O(h^{\varepsilon})$ for every $\varepsilon>0$ (this can be proved using theorem 316 of HARDY and WRIGHT [6]). Further more,

$$T_{13}=O\left(\sum_{h\leq t}k^{\omega(h)}h^{l/k}\right)=O\left(\sum_{h\leq t}h^{l/k+\varepsilon}\right)=O(t^{1+l/k+\varepsilon}).$$

Hence from (2.3) and our estimates for T_2, T_{11}, T_{12}, T_{13}, we have

$$E_{k,l}(n)=O(nt^{1-l}+t^{1+l/k+\varepsilon})+\begin{cases}O(n^{1/k}t^{1-l/k^2+\varepsilon}), & \text{if } l\leq k^2\\ O(n^{1/k}), & \text{if } l>k^2.\end{cases}$$

Taking $t=n^{k/l(k+1)}$ establishes our theorem.

§ 3. *Prachar's theorem.* In this section we make some small improvements of Theorem 2.1. to enable us to prove (1.7). Using notation introduced by Cohen, let $(u,v)_*$ denote the largest common divisor d of u and v such that $(d,v/d)=1$.

Let H be a positive integer and let a be an integer such that $(a,H)\in Q_k$ and $(a,H)_*=1$. Note that if H is k-full (and $(a,H)\in Q_k$), then $(a,H)_*=1$. Let

$$E_k(x;a,H)=O_k(x;a,H)-A_{k,H}\frac{x}{H} \tag{3.1}$$

where $A_{k,H}$ is given by (2.2). It follows from [2] that $E_k(x;a,H)=O(x^{1/k})$. We prove

Theorem 3.1. *Let a, H be subject to the above conditions and let $1>\varepsilon>0$ be arbitrary. Then*

$$E_k(x; a, H) = O(x^{1/k}s^{-1/k})+O(sk^{\omega(H)}) \tag{3.2}$$

for any s, $1\leqq s\leqq H^{1-\varepsilon}$. The constants implied by the 0-notation depend only on ε.

PROOF. Let q_k be the characteristic function of the set Q_k. From the elementary and well-known fact $q_k(n)=\sum_{d^k|n}\mu(d)$, we have

$$\begin{aligned} Q_k(x; a, H) &= \sum_{\substack{n\leqq x\\ n\equiv a(H)}} q_k(n) = \sum_{\substack{d^k m\leqq x\\ d^k m\equiv a(H)}} \mu(d) = \\ &= \sum_{\substack{d^k m\leqq x\\ d^k m\equiv a(H)\\ d\leqq y}} \mu(d)+\sum_{\substack{d^k m\leqq x\\ d^k m\equiv a(H)\\ d>y}} \mu(d) = S_1+S_2, \quad \text{say,}\end{aligned} \tag{3.3}$$

where $y=x^{1/k}s^{-1/k}$. Note that if $d^k m\equiv a \pmod H$ has any solutions d, m, then $(d, H)=1$ since $(a, H)\in Q_k$ and $(a, H)_*=1$. From (2.2) we have

$$\begin{aligned} S_1 &= \sum_{d\leqq y}\sum_{\substack{m\leqq xd^{-k}\\ d^k m\equiv a(H)}} 1 = \sum_{\substack{d\leqq y\\ (d,H)=1}} \mu(d)\{xd^{-k}H^{-1}+O(1)\} = \\ &= xH^{-1}\sum_{\substack{d=1\\ (d,H)=1}}^{\infty} \mu(d)d^{-k}-xH^{-1}\sum_{\substack{d>y\\ (d,H)=1}}\mu(d)d^{-k}+O(y) = \\ &= A_{k,H}\frac{x}{H}+O(xH^{-1}y^{1-k})+O(y) = A_{k,H}\frac{x}{H}+O(x^{1/k}s^{-1/k})\end{aligned} \tag{3.4}$$

since $xH^{-1}y^{1-k}=ysH^{-1}<y=x^{1/k}s^{-1/k}$.

From (3.3) we have

$$|S_2| \leqq \sum_{m<s}\sum_{d\leqq x^{1/k}m^{-1/k}} 1, \quad d^k m\equiv a(H). \tag{3.5}$$

For each m, let $m'=m/(m, H)$, $H'=H/(m, H)$. If $d^k m\equiv a \pmod H$ has any solutions d, then $(d, H)=1$ and $a'=a/(m, H)$ is an integer with $d^k m'\equiv a' \pmod{H'}$. The number of solutions of this latter congruence is at most $2k^{\omega(H')}\leqq 2k^{\omega(H)}$, using Evelyn and Linfoot [4], lemmas 2.41, 2.42, 2.43. Hence from (3.5)

$$\begin{aligned} |S_2| &\leqq \sum_{m<s} 2k^{\omega(H)}\{x^{1/k}m^{-1/k}(H')^{-1}+O(1)\} = \\ &= 2k^{\omega(H)}x^{1/k}H^{-1}\sum_{m<s}(m, H)m^{-1/k}+O(sk^{\omega(H)}).\end{aligned} \tag{3.6}$$

Now

$$\begin{aligned} \sum_{m<s}(m, H)m^{-1/k} &\leqq \sum_{\substack{m_1 m_2<s\\ m_1|H}} m_1(m_1 m_2)^{-1/k} = \sum_{m_1|H} m_1^{1-1/k}\sum_{m_2<s/m_1} m_2^{-1/k} = \\ &= O\Big(\sum_{m_1|H} s^{1-1/k}\Big) = O\big(\tau(H)s^{1-1/k}\big).\end{aligned} \tag{3.7}$$

Using the fact that $k^{\omega(H)}\tau(H)sH^{-1}\leqq k^{\omega(H)}\tau(H)H^{-\varepsilon}=O(1)$, we have from (3.6), (3.7)

$$S_2 = O(x^{1/k}s^{-1/k})+O(sk^{\omega(H)}). \tag{3.8}$$

Theorem 3.1 now follows from (3.1), (3.3), (3.4), and (3.8).

Corollary.

$$E_k(x;\, a,\, H) = \begin{cases} O\big((xk^{\omega(H)})^{1/(k+1)}\big), & \text{if} \quad x \leqq H^k \\ O(x^{1/k}H^{-1/(k+3/2)}), & \text{if} \quad x > H^k \end{cases} \tag{3.9}$$

where the implied constants in the 0*-estimates are absolute.*

PROOF. Assume $x\leqq H^k$. Then if $s=x^{1/(k+1)}k^{-\omega(H)k/(k+1)}$ we have $s<H^{k/(k+1)}$ and

$$x^{1/k}s^{-1/k} = sk^{\omega(H)} = (xk^{\omega(H)})^{1/(k+1)}.$$

If $x>H^k$, then taking $s=H^{k/(k+1)}k^{-\omega(H)}$ we have

$$sk^{\omega(H)} < x^{1/k}s^{-1/k}=x^{1/k}H^{-1/(k+1)}k^{\omega(H)/k} = O(x^{1/k}H^{-1/(k+3/2)}).$$

Remark 1. All of the above is subject to the condition $(a, H)_*=1$. In general, writing $H=H_1H_2$ with $H_1=(a, H)_*$, we have results analogous to the above but the error terms have an extra factor H_1 and the constant $A_{k,H}$ also depends on H_1. We state without proof

Theorem 3.2. *If* $(a, H)\in Q_k$ *and* $1>\varepsilon>0$, *then*

$$Q_k(x;\, a,\, H) = A_{k,a,H}\frac{x}{H}+O(H_1x^{1/k}s^{-1/k})+O(H_1sk^{\omega(H_2)})$$

for any s, $1\leqq s\leqq H^{1-\varepsilon}$ *where the implied constants in the O-estimates depend only on* ε. *We have*

$$A_{k,a,H} = \prod_{p|H_1}(1-p^{v_p-k})/\zeta(k)\prod_{p|H}(1-p^{-k}) \tag{3.10}$$

where $p^{v_p}\|H_1$.

COHEN and ROBINSON [2] have a proof of Theorem 3.2 with error term $O(x^{1/k})$.

Remark 2. HOOLEY ([7], Theorem 3) has a sharper result than those of this section for $k=2$. However, it does not seem to be useful in further improving the estimate of any $E_{k,l}(n)$.

§ 4. *The case* $l\leqq k^2$. In this section we establish the estimate (1.7). We shall make use of part of the proof in § 2, in particular (2.3), (2.4), and (2.5). Now

$$\begin{aligned} T_1 &= \sum_{h<t}\mu(h)Q_k(n;\, n,\, h^l) = \\ &= \sum_{h<t}\mu(h)A_{k,h^l}\frac{n}{h^l}+\sum_{h<n^{1/kl}}\mu(h)E_k(n;\, n,\, h^l)+\sum_{n^{1/kl}\leqq h<t}\mu(h)E_k(n;\, n,\, h^l) = \\ &= T_{11}+T_{12}'+T_{13}', \quad \text{say.} \end{aligned} \tag{4.1}$$

By the corollary in § 3 we have

$$T'_{12} = O\Big(\sum_{h<n^{1/kl}} n^{1/k}h^{-l/(k+3/2)}\Big) =$$

$$(4.2)\qquad = \begin{cases} O(n^{(kl+k+l/2+3/2)/kl(k+3/2)}), & \text{if } l < k+2 \\ O(n^{1/k}), & \text{if } l \geqq k+2 \end{cases} =$$

By the same corollary we have

$$T'_{13} = O\Big(\sum_{h<t} (nk^{\omega(h)})^{1/(k+1)}\Big) = O\Big(n^{1/(k+1)} \sum_{h<t} (2^{\omega(h)})^{\log_2 k^{1/(k+1)}}\Big) =$$

$$(4.3)\qquad = O\Big(n^{1/(k+1)} \sum_{h<t} (\tau(h))^{\log_2 k^{1/(k+1)}}\Big) = O\big(n^{1/(k+1)} t(\log n)^{(k^{1/(k+1)}-1)}\big),$$

$$(4.3)\qquad = O\Big(n^{1/(k+1)} \sum_{h<t} (\tau(h))^{\log_2 k^{1/(k+1)}}\Big) = O\big(n^{1/(k+1)} t(\log n)^{(k^{1/(k+1)}-1)}\big),$$

where we use a formula of Ramanujan (see WILSON [13], eq. (2.39)).

It now follows from (2.3), (2.4), (2.5), (4.1), (4.2), (4.3) that

$$E_{k,l}(n) = O(nt^{1-l}) + O(n^{(k+l)/l(k+1)}) + O\big(n^{1/(k+1)} t(\log n)^{(k^{1/(k+1)}-1)}\big).$$

Letting $t = n^{k/l(k+1)}(\log n)^{-(k^{1/(k+1)}-1)/l}$, we have (1.7).

§ 5. *The average of* $E_{k,l}(n)$. Let $Q_k(x)$ be the number of k-free numbers up to x and $\Delta_k(x) = Q_k(x) - x/\zeta(k)$. It is known (WALFISZ [12], Satz 1, p. 192) that

$$(5.1)\qquad \Delta_k(x) = O\big(x^{1/k}\delta_k(x)\big)$$

where $\delta_k(x) = \exp\{-Ak^{-8/5}(\log x)^{3/5}(\log\log x)^{-1/5}\}$, where A is a positive absolute constant. If the Riemann hypothesis is true then it is known ([11], corollary 3.2.1) that

$$(5.2)\qquad \Delta_k(x) = O(x^{2/(2k+1)}\omega(x))$$

where $\omega(x) = \exp\{A \log x\,(\log\log x)^{-1}\}$.

Let $S_{k,l}(x) = \sum_{a+b \leqq x} q_k(a)q_l(b)$. It follows from (5.1) and partial summation that

$$(5.3)\qquad S_{k,l}(x) = x^2/2\zeta(k)\zeta(l) + O\big(x^{1+1/k}\delta_k(x)\big).$$

If the Riemann hypothesis is true it follows from (5.2) and partial summation that

$$(5.4)\qquad S_{k,l}(x) = x^2/2\zeta(k)\zeta(l) + O\big(x^{1+2/(2k+1)}\omega(x)\big).$$

However, it is clear that $S_{k,l}(x) = \sum_{n \leqq x} T_{k,l}(n)$, so that from (1.3) we have

$$(5.5)\qquad \sum_{n\leqq x} E_{k,l}(n) = S_{k,l}(x) - \sum_{n \leqq x} c_{k,l}\varrho_{k,l}(n)n.$$

We now prove

Lemma.

$$\sum_{n\leqq x} c_{k,l}\varrho_{k,l}(n)n = x^2/2\zeta(k)\zeta(l) + O(x).$$

PROOF. Define a multiplicative function h by letting $h(1)=1$ and $h(p^a)=p^l-p^{l-k}-1$ for primes p and all integers $a\geqq 1$. Then it is easy to verify that

$$\varrho_{k,l}(n)=\sum_{d^k|n}\mu^2(d)/h(d).$$

Hence

$$\begin{aligned}\sum_{n\leqq x}\varrho_{k,l}(n)n &= \sum_{n\leqq x} n\sum_{d^k|n}\mu^2(d)/h(d)=\\ &=\sum_{d^k m\leqq x} d^k m\mu^2(d)/h(d)=\sum_{d\leqq x^{1/k}}\frac{d^k\mu^2(d)}{h(d)}\sum_{m<x/d^k} m=\\ &=\sum_{d\leqq x^{1/k}}\frac{d^k\mu^2(d)}{h(d)}\left\{\frac{x^2}{2d^{2k}}+O\left(\frac{x}{d^k}\right)\right\}=\\ &=\frac{x^2}{2}\sum_{d=1}^{\infty}\frac{\mu^2(d)}{d^k h(d)}+O\big(x^2\sum_{d>x^{1/k}}\mu^2(d)/d^k h(d)+O\big(x\sum_{d\leqq x^{1/k}}\mu^2(d)/h(d)\big).\end{aligned}\tag{5.6}$$

But $h(p)\geqq p^l-p^{l-2}-1\geqq p^l/2$, so that if d is square free, $h(d)\geqq d^l 2^{-\omega(d)}$. Hence

$$\sum_{d>x^{1/k}}\mu^2(d)/d^k h(d)=O(x^{(-k-l+1+\varepsilon)/k})=O(x^{-1}),$$

$$\sum_{d\leqq x^{1/k}}\mu^2(d)/h(d)=O(1).$$

Hence it follows from (5.6) that

$$\sum_{n\leqq x} c_{k,l}\varrho_{k,l}(n)n=\frac{c_{k,l}x^2}{2}\sum_{d=1}^{\infty}\frac{\mu^2(d)}{d^k h(d)}+O(x).\tag{5.7}$$

From (1.4) we have $E_{k,l}(n)=o(n)$, so $\sum_{n\leqq x}E_{k,l}(n)=o(x^2)$. Hence from (5.3) and (5.5), we have

$$\sum_{n\leqq x} c_{k,l}\varrho_{k,l}(n)n\sim x^2/2\zeta(k)\zeta(l).\tag{5.8}$$

The lemma now follows from (5.7) and (5.8).

Theorem 5.1. *We have*

$$\frac{1}{x}\sum_{n\leqq x}E_{k,l}(n)=O\big(x^{1/k}\delta_k(x)\big).\tag{5.9}$$

If the Riemann hypothesis is true, we have

$$\frac{1}{x}\sum_{n\leqq x}E_{k,l}(n)=O\big(x^{2/(2k+1)}\omega(x)\big).\tag{5.10}$$

PROOF. The theorem follows from (5.3), (5.4), (5.5) and the lemma.

Remark. We note that if $l<k^2$, the average error as given by (5.9) is considerably smaller than the error estimates (1.6) or (1.7). But if $l\geqq k^2$, the average error is not much smaller than the error estimate (1.6). The estimate (5.10) suggests that if the Riemann hypothesis is true, then there may be further improvements possible in the estimation of the $E_{k,l}(n)$.

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