On certain types of Kähler spaces

By M. C. CHAKI and A. K. RAY (Calcutta)

(Dedicated to Prof. Ram Behari on his 80th birthday)

Introduction. A non-flat n-dimensional Riemannian space V_n in which the curvature tensor R_{kjih} satisfies a relation of the form

(1) $$\nabla_m \nabla_l R_{kjih} = \beta_m \nabla_l R_{kjih} + a_{lm} R_{kjih}$$

where β_m and a_{lm} are not both zero, has been called a generalised 2-recurrent space or briefly a G 2-recurrent space [1] and has been studied in some details [2, 3, 4]. A V_n in which the conformal curvature tensor C_{kjih} satisfies a relation of the type (1) has been called a generalised conformaly 2-recurrent space [1]. It may be briefly called a GC 2-recurrent space. The present paper deals with an n-dimensional ($n=2N$, $N\neq 1, 2$) Kähler space. In the first part of the paper, consisting of sections 2—5, canonical representations of the relevant tensors have been obtained for a G 2-recurrent Kähler space and as a consequence it has been shown that such a space is necessarily a recurrent space. The second part (Section 6) deals with a GC 2-recurrent Kähler space. It has been shown that such a space reduces to a G 2-recurrent space and is therefore a recurrent Kähler space (β_m and a_{lm} are called the vector and tensor of recurrence. It is assumed throughout that a_{lm} is different from zero)

1. *Preliminaries.* Let F_i^h be the structure tensor and g_{ij} the positive definite Riemannian metric of a Kähler space in real representation. Then

$$F_j^r F_r^i = -\delta_j^i \quad \text{and} \quad g_{rt} F_j^r F_i^t = g_{ji}$$

It is also known that

(1.1) $$F_{ji} = g_{ri} F_j^r = -F_{ij}, \quad F^{ji} = g^{jr} F_r^i = -F^{ij}, \quad \nabla_k F_{ji} = 0.$$

Let R_{ij} be the Ricci tensor and $R=g^{ij}R_{ij}$ the scalar curvature. Also let $H_{ij}= =\frac{1}{2} R_{ijkl} F^{kl}$. Then the following relations hold [5]:

(1.2) $$H_{ij} = -H_{ji}$$ (1.3) $$R_{ks} F_j^s = H_{kj}$$

(1.4) $$H_{ks} F_j^s = -R_{kj}$$ (1.5) $$H_{kj} F^{kj} = -R.$$

Part I. — G 2-recurrent Kähler space

2. *Some useful relations in G 2-recurrent Kähler space.* Let the defining relation of the space be

(2.1) $$\nabla_m \nabla_l R_{kjih} = \beta_m \nabla_l R_{kjih} + a_{lm} R_{kjih}.$$

From the Bianchi identity:

$$\nabla_l R_{kjih}+\nabla_j R_{lkih}+\nabla_k R_{jlih}=0$$

we have

$$\nabla_m \nabla_l R_{kjih}+\nabla_m \nabla_j R_{lkih}+\nabla_m \nabla_k R_{jlih}=0.$$

In virtue of (2.1) this gives

$$a_{lm} R_{kjih}+a_{jm} R_{lkih}+a_{km} R_{jlih}=0. \tag{2.2}$$

(The β terms cancel out.) Replacing m by s and transvecting with $a_m{}^s=g^{rs}a_{mr}$ we get

$$b_{lm} R_{kjih}+b_{jm} R_{lkih}+b_{km} R_{jlih}=0, \tag{2.3}$$

where

$$b_{lm}=a_{ls}a_m{}^s=a_{ls}a_{mt}g^{st}.$$

It may be noted that b_{lm} is a symmetric tensor. Let $\Theta=g^{lm}b_{lm}$. Then $\Theta=a^{mt}a_{mt}\neq 0$, for otherwise a_{ij} would vanish identically. This shows that $b_{ij}\neq 0$.
We may now quickly deduce the following results:

$$R_{jli}{}^r b_{rm}=R_{il}b_{jm}-R_{ij}b_{lm} \tag{2.4}$$

$$R_{rj}b^r_m=\frac{1}{2}Rb_{jm} \tag{2.5}$$

$$\Theta R_{kjih}=R_{hk}b_{ij}+R_{ij}b_{hk}-R_{hj}b_{ik}-R_{ik}b_{hj}. \tag{2.6}$$

In fact, (2.4) may be obtained from (2.3) by contracting with g^{hk}, (2.5) follows from (2.4) on contraction with g^{il} and (2.6) may be obtained by contracting (2.3) with g^{lm} and using (2.4). Now, we have

$$\begin{aligned}2\Theta H_{kj} &= \Theta R_{kjih}F^{ih}=\\ &= R_{hk}F^{ih}b_{ij}+R_{ij}F^{ih}b_{hk}-R_{hj}F^{ih}b_{ik}-R_{ik}F^{ih}b_{hj}=\\ &= H_{ks}g^{is}b_{ij}-H_{js}g^{hs}b_{hk}-H_{js}g^{is}b_{ik}+H_{ks}g^{hs}b_{hj}=\\ &= 2(H_{ks}b_j{}^s-H_{js}b_k{}^s).\end{aligned}$$

Therefore

$$\Theta H_{kj}=H_{ks}b_j{}^s-H_{js}b_k{}^s. \tag{2.7}$$

Transvecting (2.5) with $F_p{}^j$ we get $H_{rp}b^r_m=\frac{1}{2}Rb_{jm}F_p{}^j=\frac{1}{2}Rb_{ms}F_p{}^s$. Hence (2.7) gives

$$\Theta H_{kj}=\frac{1}{2}R(b_{ks}F_j{}^s-b_{js}F_k{}^s). \tag{2.8}$$

Transvecting this with $F_p{}^j$ we get

$$\Theta R_{kp}=\frac{1}{2}R(b_{kp}+b^*_{kp}) \tag{2.9}$$

where

$$b^*_{kp}=b_{sj}F_k{}^sF_p{}^j. \tag{2.10}$$

It is to be noted that symmetry of b_{kp} implies that b^*_{kp} is also symmetric.

3. *Form of R_{ij} in G 2-recurrent Kähler Space*

From the relation (2.5) viz.,

$$R_{rj} b^r_m = \frac{1}{2} R b_{jm} \tag{3.1}$$

we see that the columns and therefore the rows of b_{ij} define a set of vectors which are all eigenvectors of R_{ij} corresponding to the root $\frac{1}{2} R$ (Such a set depends upon the coordinate system. Under a different coordinate system we shall get a different set). If $R=0$ then from (2.9) we get $R_{ij}=0$ (because $\Theta \neq 0$) whence by (2.6) $R_{kjih}=0$. But our space is by definition non-flat. Hence R cannot be zero. Also the relation $\nabla_l \nabla_m R = \beta_m \nabla_l R + a_{lm} R$, obtained from (2.1), shows that if in a G 2-recurrent space R is nonzero, then it is non-constant. Hence we obtain

Lemma 1. *In a G 2-recurrent Kähler space the scalar curvature is non-zero and non-constant.*

Let m be the dimension of the eigensubspace of R_{ij} corresponding to the root $\frac{1}{2} R$ and let $\underset{1}{u_i}, \ldots\ldots \underset{m}{u_i}$ be an orthonormalised set of vectors spanning this subspace. Then evidently b_{ij} will be of the form

$$b_{ij} = \underset{1}{\lambda_i} \underset{1}{u_j} + \underset{2}{\lambda_i} \underset{2}{u_j} + \ldots + \underset{m}{\lambda_i} \underset{m}{u_j} = \sum_{p=1}^{m} \underset{p}{\lambda_i} \underset{p}{u_j} \tag{$*$}$$

where λ's define another set of m vectors. Symmetry of b_{ij} requires

$$\sum_{p=1}^{m} (\underset{p}{\lambda_i} \underset{p}{u_j} - \underset{p}{\lambda_j} \underset{p}{u_i}) = \sum_{p=1}^{m} (\underset{p}{\lambda} \wedge \underset{p}{u})_{ij} = 0,$$

where $\wedge$ denotes exterior product. By Cartan's lemma ([6], p. 18), we then have

$$\underset{p}{\lambda_i} = \sum_{q=1}^{m} d_{pq} u_i, \tag{$*'$}$$

where d_{pq} is an m-ordered symmetric matrix. From ($*$) and ($*'$) we have

$$b_{ij} = \sum_{p=1}^{m} \sum_{q=1}^{m} d_{pq} \underset{p}{u_i} \underset{q}{u_j}. \tag{3.2}$$

It is clear that rank d_{pq} = rank $b_{ij} = l$, say. (Indices like p, q, denoting primarily collection of objects, have no tensorial significance. Summation over any such index will be explicitly denoted by $\sum$ notation). Now,

$$b^*_{ij} = b_{kl} F^k_i F^l_j = \sum_{p=1}^{m} \sum_{q=1}^{m} d_{pq} \underset{p}{u_k} \underset{q}{u_l} F^k_i F^l_j = \sum_{p=1}^{m} \sum_{q=1}^{m} d_{pq} \underset{p}{V_i} \underset{q}{V_j} \tag{$*''$}$$

where

$$\underset{p}{V_i} = \underset{p}{u_k} F^k_i \quad (p = 1, 2, \ldots, m).$$

Let for any vector u, the vector V given by $V_i = u_k F_i^k$ be called the associate vector of u with respect to the structure tensor F. Then

$$V_k = g_{it} u^t F_k^{\ i} = F_i^{\ l} F_t^{\ m} g_{lm} u^t F_k^{\ i} = -g_{km} u^t F_t^{\ m} u^t$$

whence

$$(*''') \qquad V^j = -F_t^{\ j} u^t.$$

We now prove a simple result in the form of

Lemma 2. *If u_i is an eigenvector of R_{ij} corresponding to a characteristic root ϱ, then V_i, the associate vector of u_i, is also an eigenvector corresponding to the same root.*

Let

$$(3.3) \qquad R_{ij} u^j = \varrho u_i.$$

From (1.3) and (1.4) we have $R_{ij} = R_{kl} F_i^k F_j^{\ l}$. Therefore (3.3) can be written as $R_{kl} F_i^k F_j^{\ l} u^j = \varrho u_i$. In virtue of $(*''')$ this gives $R_{kl} F_i^k (-V^l) = \varrho u_i$. Transvecting with $F_s^{\ i}$ we get $R_{kl}(-\delta_s^{\ k})(-V^l) = \varrho V_s$ whence $R_{sl} V^l = \varrho V_s$. This proves the lemma.

In virtue of this result, we see that the vectors $\underset{1}{V_i}, \ldots\ldots \underset{m}{V_i}$ also all belong to the eigensubspace of R_{ij} corresponding to the root $\frac{1}{2} R$ and therefore these vectors are all linear combinations of the u-vectors.
Hence b^*_{ij} is eventually of the form

$$(3.4) \qquad \sum_{p=1}^{m} \sum_{q=1}^{m} d^*_{pq} \underset{p}{u_i} \underset{q}{u_j}$$

where d^*_{pq} is also symmetric. Since F is non-singular, it may be seen that $\operatorname{rank} b_{ij} = \operatorname{rank} b^*_{ij}$. Hence $\operatorname{rank} d^*_{pq} = \operatorname{rank} d_{pq} = l$. From (2.9), (3.2) and (3.4) we obtain

$$(3.5) \qquad \Theta R_{ij} = \frac{1}{2} R \sum_{p=1}^{m} \sum_{q=1}^{m} D_{pq} \underset{p}{u_i} \underset{q}{u_j}$$

where

$$D_{pq} = d_{pq} + d^*_{pq}.$$

Thus D_{pq} is also symmetric in p and q.
Since every u-vector is an eigenvector of R_{ij} corresponding to the root $\frac{1}{2} R$, we have

$$(3.6) \qquad R_{ir} \underset{s}{u^r} = \frac{1}{2} R \underset{s}{u_i} \quad (s = 1, \ldots, m).$$

From (3.5) and (3.6) we get

$$\Theta R_{ir} u^r_s = \frac{1}{2} R \sum_{p=1}^{m} \sum_{q=1}^{m} D_{pq} \underset{p}{u_i} \underset{q}{u_r} \underset{s}{u^r} = \Theta \frac{1}{2} R \underset{s}{u_i},$$

or

$$\langle \underset{q}{u}, \underset{s}{u} \rangle \left(\sum_{p=1}^{m} D_{pq} \underset{p}{u_i} \right) = \Theta \underset{s}{u_i},$$

where

$$\langle \underset{q}{u}, \underset{s}{u} \rangle = \underset{q}{u}_r \underset{s}{u}^r = \delta_{rs}$$

in virtue of the orthonormality of the vectors $\underset{s}{u}$. Hence

$$\sum_{p=1}^{m} D_{ps} \underset{p}{u}_i = \Theta \underset{s}{u}_i.$$

Since the u-vectors are independent, we obtain

$$\text{(3.7)} \qquad \begin{cases} D_{ps} = 0 & \text{for} \quad p \neq s \\ D_{ss} = \Theta & (s = 1, \ldots, m). \end{cases}$$

This gives $\Theta R_{ij} = \frac{1}{2} R\Theta (\underset{1}{u}_i \underset{1}{u}_j + \ldots + \underset{m}{u}_i \underset{m}{u}_j)$. Since $\Theta \neq 0$, we have

$$\text{(3.8)} \qquad R_{ij} = \frac{1}{2} R(\underset{1}{u}_i \underset{1}{u}_j + \ldots + \underset{m}{u}_i \underset{m}{u}_j).$$

Raising i and contracting with j, we get $R = \dfrac{m}{2} R$ showing that m is precisely equal to 2. Writing u_i and V_i for $\underset{1}{u}_i$ and $\underset{2}{u}_i$ we get

$$\text{(3.9)} \qquad R_{ij} = \frac{1}{2} R(u_i u_j + V_i V_j).$$

The eigensubspace of R_{ij} corresponding to the root $\frac{1}{2} R$ is thus of dimension 2 and coincides with the rowspace (Column space) of R_{ij}. Hence rank $R_{ij} = 2$. In fact, from (3.9) we get $R_{ik} R_j^{\ k} = \frac{1}{2} R R_{ij}$. Hence the minimal equation of R_{ij} is $\varrho^2 - \frac{1}{2} R\varrho = 0$ implying that the characteristic roots are $\frac{1}{2} R, \frac{1}{2} R, 0, 0, \ldots, 0\,(n-2$ zeros). It may further be noted that u_i may be chosen arbitrarily in the eigensubspace of R_{ij} corresponding to the root $\frac{1}{2} R$ and since V_i is in the same eigensubspace and orthogonal to u_i, we have either $V_i = u_k F_i^k$ or $V_i = -u_k F_i^k$. For convenience we choose $V_i = u_k F_i^k$.

4. *Forms of H_{ij} and R_{kjih}.*

In virtue of (3.9) we have

$$\text{(4.1)} \qquad H_{ij} = R_{ik} F_j^k = \frac{1}{2} R(u_i u_k + V_i V_k) F_j^k = \frac{1}{2} R(u_i V_j - V_i u_j) = \frac{1}{2} R A_{ij}$$

where

$$A_{ij} = u_i V_j - V_i u_j.$$

We next observe that d_{pq} is a 2-ordered symmetric matrix. Therefore b_{ij} is of the form

$$\text{(4.2)} \qquad b_{ij} = \lambda u_i u_j + \varrho (u_i V_j + V_i u_j) + \mu V_i V_j$$

where $\lambda+\mu=\Theta$ and the rank of the matrix $\begin{bmatrix}\lambda & \varrho \\ \varrho & \mu\end{bmatrix}$ is 2 or 1. Of course, in the latter case by choosing u_i appropriately we may have

(4.3) $$b_{ij} = \Theta u_i u_j.$$

In either case by straightforward calculation we obtain from (2.6)

(4.4) $$R_{kjih} = -\frac{1}{2} RA_{kj} A_{ih}$$

where

$$A_{ij} = u_i V_j - V_i u_j.$$

5. *A G 2-recurrent Kähler space is a recurrent space*

Since $A_{ih}A^{ih}=2$ we see that

(5.1) $$R^{kjih}R_{kjih} = \left(-\frac{1}{2} RA^{kj}A^{ih}\right)\left(-\frac{1}{2} RA_{kj}A_{ih}\right) = R^2.$$

As an immediate consequence of (5.1) and the defining relation (2.1) of the space we have

(5.2) $$\nabla_l R^{kjih} \nabla_m R_{kjih} = \nabla_l R \nabla_m R.$$

Let

(5.3) $$S_{kjihl} = \nabla_l R_{kjih} - \lambda_l R_{kjih}$$

where

(5.4) $$\lambda_l = \frac{1}{R} \nabla_l R.$$

Then

(5.5) $$\begin{aligned} S^{kjihl} S_{kjihl} &= \lambda^l \lambda_l R^{kjih} R_{kjih} - \lambda^p R_{kjih} \nabla_p R^{kjih} \\ &\quad - \lambda^l R^{kjih} \nabla_l R_{kjih} + g^{lp} \nabla_l R_{kjih} \nabla_p R^{kjih}. \end{aligned}$$

In virtue of (5.1) and (5.2) the righthand side of (5.5) can be expressed as

$$\lambda^l \lambda_l R^2 - 2R\lambda^p \nabla_p R + g^{lp} \nabla_l R \nabla_p R.$$

But this becomes zero because of (5.4). Hence $S^{kjihl}S_{kjihl}=0$. Since the metric of the space is positive definite we get $S_{kjihl}=0$. Hence $\nabla_l R_{kjih}=\lambda_l R_{kjih}$. That is, the space is recurrent. We summarise the main results in the form of

Theorem 1. *A G 2-recurrent Kähler space is a recurrent Kähler space and is necessarily of non-vanishing (therefore non-constant) scalar curvature. The tensors* R_{ij}, H_{ij} *and* R_{kjih} *may be simultaneously rendered the canonical forms given by* (3.9), (4.1) *and* (4.4) *where* u_i *and* V_i *are mutually orthogonal unit vectors spanning the eigen subspace of* R_{ij} *corresponding to the root* $\frac{1}{2}R$.

Putting $\beta=0$ throughout, we obtain the following corollary: A 2-recurrent Kähler space is a recurrent Kähler space (Other statements of Theorem 1 are also equally valid for this space)

Part II — A *GC* 2-recurrent Kähler space

6. *A GC 2-recurrent Kähler space is a G 2-recurrent Kähler space*

We now consider a non-flat n-dimensional ($n=2N$, $N\neq 1, 2$) Kähler space in which Weyl's conformal curvature tensor C_{kjih} satisfies the relation

$$\nabla_m \nabla_l C_{kjih} = \beta_m \nabla_l C_{kjih} + a_{lm} C_{kjih} \tag{6.1}$$

where a_{lm} is not zero. Now,

$$C_{kjih} \overset{\text{def}}{=} R_{kjih} - \frac{1}{n-2}[g_{kh}R_{ji} + g_{ji}R_{kh} - g_{ki}R_{jh} - g_{jh}R_{ki}] + \frac{R}{(n-1)(n-2)}[g_{kh}g_{ji} - g_{ki}g_{jh}].$$

This gives

$$\begin{aligned} C_{kjih}F^{ih} &= 2H_{kj} + \frac{1}{n-2}[R_{ji}F_k^{\ i} - R_{kh}F_j^{\ h} + R_{jh}F_k^{\ h} - R_{ki}F_j^{\ i}] - \\ &\quad - \frac{R}{(n-1)(n-2)}[g_{ji}F_k^{\ i} + g_{jh}F_k^{\ h}] = \\ &= 2H_{kj} + \frac{1}{n-2}[-4H_{kj}] - \frac{R}{(n-1)(n-2)}[2F_{kj}] = \\ &= \frac{2(n-4)}{n-2}H_{kj} - \frac{2R}{(n-1)(n-2)}F_{kj}. \end{aligned} \tag{6.2}$$

Since covariant derivative of F_{kj} vanishes identically, from (6.1) and (6.2) we have

$$\begin{aligned} &\frac{2(n-4)}{n-2}[\nabla_m \nabla_l H_{kj} - \beta_m \nabla_l H_{kj} - a_{lm} H_{kj}] + \\ &- \frac{2R}{(n-1)(n-2)}[\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R] F_{kj} = 0. \end{aligned} \tag{6.3}$$

Transvecting this with F^{kj} we have

$$-\frac{2(n-4)}{n-2}[\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R] - \frac{2(n-2)}{n-1}[\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R] = 0$$

or,

$$-\frac{2(n-2)}{n-1}[\nabla_m \nabla_l R - \beta_m \nabla_l - a_{lm} R] = 0.$$

Hence (6.4)

$$\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R = 0$$

whence from (6.3) we get

$$\nabla_m \nabla_l H_{kj} - \beta_m \nabla_l H_{kj} - a_{lm} H_{kj} = 0. \tag{6.5}$$

Transvecting this with $F_p^{\ j}$ and then writing j in place of p we get

$$\nabla_m \nabla_l R_{kj} - \beta_m \nabla_l R_{kj} - a_{lm} R_{kj} = 0. \tag{6.6}$$

In virtue of (6.1), (6.4) and (6.6) we now obtain

$$\nabla_m \nabla_l R_{kjih} - \beta_m \nabla_l R_{kjih} - a_{lm} R_{kjih} = 0, \tag{6.7}$$

i.e. the space is a G 2-recurrent space. Thus we arrive at the following theorem:

Theorem 2. *A GC 2-recurrent Kähler space is a G 2-recurrent Kähler space and is therefore a recurrent Kähler space.*

Corollary: *A C 2-recurrent (conformally 2-recurrent) Kähler space is a 2-recurrent Kähler space and is therefore a recurrent Kähler space.*

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DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA
AND
DÉPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI

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