A theorem on error bounds in $L_2(0, 2\pi)$ space

By P. GOYALIYA (Calgary, Alberta)

1. JACKSON, D. [2] obtained the error bounds of functions having modulus of continuity, by means of trigonometric polynomials in $L(0, 2\pi)$ space. He also discussed the same aspect for functions having one or more derivatives. The problem of error bounds in $L_2(0, 2\pi)$ space has been studied by N. I. CERNYH [1]. In this note we have obtained the error bounds of functions having one or more derivatives in $L_2(0, 2\pi)$ space.

2. Let $L_2(0, 2\pi)$ be the space of measurable functions $f(x)$ for which $\int\limits_0^{2\pi} |f(x)|^2\,dx<\infty$. Let

$$\|f(x)\|_{L_2} = \left\{\frac{1}{\pi}\int\limits_0^{2\pi} |f(x)|^2\,dx\right\}^{1/2}, \tag{2.1}$$

$$w(\delta, f)_{L_2} = \sup_{|t|\leqq\delta} \|f(x+t)-f(x)\|_{L_2} \tag{2.2}$$

and

$$E_n(f)_{L_2} = \|f(x)-s_n(x; f)\|_{L_2} \tag{2.3}$$

where $s_n(x, f)$ are the partial sums of order n of the Fourier series of $f(x)$.

Throughout the note, M, without suffixes denotes a positive constant, not necessarily the same at each occurrence.

3. On Jackson's inequality in $L_2(0, 2\pi)$ space N. I. CERNYH [1] proved the following Theorem.

Theorem A. *Let* $f(x)\in L_2(0, 2\pi)$, $f(x)\neq constant$. *For* $n=0, 1, 2, \ldots$ *we have*

$$E_n(f)_{L_2} \leqq \frac{1}{\sqrt{2}}\, w\left(\frac{\pi}{n+1}, f\right)_{L_2}. \tag{3.1}$$

We generalize the above Theorem in the following form.

Theorem 1. $f^{(m)}(x)\in L_2(0, 2\pi)$ $m\geqq 0$, $f^{(m)}(x)\neq constant$. *For* $n=0, 1, 2, \ldots,$ *we have*

$$E_n(f^{(m)})_{L_2} \leqq \frac{1}{\sqrt{2}}\, w\left(\frac{\pi}{n+1}, f^{(m)}\right)_{L_2}, \tag{3.2}$$

and

$$E_n(f)_{L_2} \leqq \frac{M}{n^m}\, E_n(f^{(m)})_{L_2}. \tag{3.3}$$

It is interesting to note that for $m=0$, Theorem 1 reduces to Theorem A.
4. We require the following lemma for the proof of our Theorem.

Lemma 1. *Let $N \geqq 1$ be an integer, let $l_k (k \geqq N)$ be real no's and $0 < \sum l_k^2 < \infty$, then the function*

$$F_N(t) = \sum_{k=N}^{\infty} l_k^2 \cos kt \tag{4.1}$$

takes both positive and negative values on $(0, \pi/N)$.

Proof of Lemma. Put

$$\varphi_N(t) = \begin{cases} -\sin Nt & \text{for} \quad 0 \leqq t \leqq \dfrac{\pi}{N}, \\ 0 & \text{for} \quad \dfrac{\pi}{N} \leqq t \leqq \pi. \end{cases}$$

Expanding this function in a Fourier series of the form

$$\varphi_N(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos kt,$$

we easily see that $A_N = 0$ and

$$A_k = \frac{4N}{\pi} \frac{1}{k^2 - N^2} \cos^2 \frac{k\pi}{2N} \geqq 0 \quad (k > N).$$

From this we obtain by Parseval's equation

$$\frac{1}{\pi} \int_0^{\pi} \varphi_N(t) F_n(t)\, dt = \sum_{k=N}^{\infty} l_k^2 A_k \geqq 0.$$

If we suppose that $F_n(t) \geqq 0$ for $0 < t < \dfrac{\pi}{N}$ then, by the definition of $\varphi_n(t)$ and the fact that $F_n(t) > 0$ in a neighbourhood of $t=0$, we obtain the inequality

$$\int_0^{\pi} \varphi_n(t) F_n(t)\, dt < 0$$

which contradicts the preceding one. This establishes the lemma.

5. Proof of Theorem 1.

Let
$$f(x) \sim \sum_1^{\infty} (a_k \cos kx + b_k \sin kx)$$
then
$$f^{(m)}(x) \sim \sum_1^{\infty} k^m (a_k \cos kx + b_k \sin kx).$$

Now

$$E_n^2(f^{(m)}) = \|f^{(m)}(x) - S_n(x, f^{(m)})\|^2 =$$

(5.1)

$$= \Big\| \sum_{n+1}^{\infty} k^m (a_k \cos kx + b_k \sin kx) \Big\|^2 = \sum_{n+1}^{\infty} k^{2m}(a_k^2 + b_k^2),$$

by Parseval's inequality.

Also

(5.2) $$\|f^{(m)}(x+t) - f^{(m)}(x)\|_{L_2}^2 =$$

$$= \Big\| \sum_1^{\infty} k^m \big(a_k \cos k(x+t) + b_k \sin k(x+t)\big) - \sum_1^{\infty} k^m (a_k \cos kx + b_x \sin kx) \Big\|_{L_2}^2 =$$

$$= 2 \sum_1^{\infty} k^{2m}(a_k^2 + b_k^2)(1 - \cos kt)$$

by Parseval's inequality.

Since $1 - \cos kt \geqq 0$, we have, from (5.2)

$$w^2(\pi/n+1, f^{(m)})_{L_2} = 2 \sup_{|t| < \frac{\pi}{n+1}} \sum_{k=1}^{\infty} k^{2m}(a_k^2 + b_k^2)(1 - \cos kt) \geqq$$

$$\geqq 2 \sup_{|t| < \frac{\pi}{n+1}} \sum_{k=n+1}^{\infty} k^{2m}(a_k^2 + b_k^2)(1 - \cos kt) \geqq$$

$$\geqq 2 \sum_{k=n+1}^{\infty} k^{2m}(a_k^2 + b_k^2) - 2 \inf_{|t| < \frac{\pi}{n+1}} \sum_{k=n+1}^{\infty} k^{2m}(a_k^2 + b_k^2) \cos kt \geqq$$

$$\geqq 2 \sum_{k=n+1}^{\infty} k^{2m}(a_k^2 + b_k^2),$$

by Lemma 1 $\big(\inf \sum k^{2m}(a_k^2 + b_k^2) \cos kt \leqq 0\big)$.

Now we have

$$2E_n^2(f^{(m)})_{L_2} = 2 \sum_{k=n+1}^{\infty} k^{2m}(a_k^2 + b_k^2) \leqq w^2\left(\frac{\pi}{n+1}, f^{(m)}\right)_{L_2}$$

$\big($by (5.1)$\big)$, or

$$E_n^2(f^{(m)})_{L_2} \leqq \frac{1}{\sqrt{2}} w^2\left(\frac{\pi}{n+1}, f^{(m)}\right)_{L_2}.$$

Now, to prove (3.3) we have

$$E_n^2(f) = \|f(x) - S_n(x,f)\|^2 = \Big\| \sum_{n+1}^{\infty} (a_k \cos kx + b_k \sin kx) \Big\|^2 = \sum_{n+1}^{\infty} (a_k^2 + b_k^2)$$

by Parseval's inequality

$$\leqq \frac{M}{n^{2m}} \sum_{n+1}^{\infty} k^{2m}(a_k^2+b_k^2) = \frac{M}{n^{2m}} E_n^2(f^{(m)}) \quad (M \text{ is a constant})$$

i.e.

$$E_n(f)_{L_2} \leqq \frac{M}{n^m} E_n(f^{(m)})_{L_2}.$$

This proves Theorem 1.

6. It may be interesting to know the answers to the following questions:

(i) Can Theorem 1 be extended to $(c, 1)$ summability, or Matrix summability?

(ii) Can the result be extended to some other series, viz. Legendre series, ultraspherical series, Bessel's series, etc.?

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References

[1] N. I. Cernyh, On Jackson's inequality in L_2. *Proc. of the Steklov Institute of Math.* **88** (1967), 71—74.

[2] D. Jackson, The Theory of approximation, *Amer. Math. Soc. Coll. Publ.* Vol. XI (1930).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALGARY
CALGARY, ALBERTA, CANADA

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