# On the orthonormal frame bundle of a Riemannian manifold

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## 1 §. Introduction

There are more possibilities to introduce Riemannian metrics on the tangent bundle or on the tangent sphere bundle of a Riemannian manifold. One of the most natural metrics is introduced by S. SASAKI in [4]. In recent papers KLINGENBERG and SASAKI [1] and the author [2] investigated the tangent sphere bundle of a Riemannian 2-sphere and in generally of a 2-manifold, respectively.

However, the tangent sphere bundle of an oriented 2-manifold is isomorphic to its orthonormal frame bundle. So, it is a natural question: How could we define a Riemannian metric on the orthonormal frame bundle of a Riemannian manifold analogously to the Sasaki metric, and how could we generalize the results in [1] and [2]. The purpose of the present paper is to answer for these questions.

We shall characterize the geodesics on $O(M)$ in § 4.

In § 5, we consider the case when the basic manifold is locally symmetric.

In § 6, we investigate the geometry of the orthonormal frame bundle of an $n$-sphere.

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## 2 §. Preliminaries

Let $R^n$ be the linear euclidean space and $V^n$ denote the manifold of orthonormal $n$-frames in $R^n$. The orthogonal group $O(n)$ acts simply transitively on $V^n$. It is possible to define a Riemannian metric on $V^n$, which is invariant with respect to the action of $O(n)$ on $V^n$, in a natural manner. The construction may be described as follows.

Let $z(t)=(e_1(t), \dots, e_n(t))$ be a curve on $V^n$. Let $\alpha(t)$ be a corresponding curve on $O(n)$:

$$z(t) = \alpha(t)z(t_0).$$

The Riemannian metric $\tilde{g}$ on $V^n$ at $z(t_0)$ is defined by

$$\tilde{g}(\dot{z}, \dot{z}) := \frac{1}{2}\operatorname{Trace}(\dot{\alpha}\dot{\alpha}^*),$$

where $\alpha^*$ is the transponed matrix and point denotes the derivation by $t$. Let $E_{ik}$

$(i>k)$ denote the usual basis of the orthogonal Lie algebra $o(n)$, and let $\dot{\alpha}=\sum_{i>k}\omega_{ik}E_{ik}$. Then

$$\dot{z}(t_0)=(\ldots,\dot{e}_i(t_0),\ldots)=d\alpha z(t_0)=(\ldots,\sum_{i>k}\omega_{ik}e_k,\ldots),$$

and

$$\tilde{g}(\dot{z},\dot{z})=\sum_{i>k}(\omega_{ik})^2. \tag{1}$$

Now, let $M$ be a Riemannian manifold and $g$ denote its metric tensor. We can define a family of natural Riemannian metrics on the orthonormal fram bundle $O(M)$ as follows.

Let $X$, $Y$ be tangent vectors at $u\in O(M)$, and $x=p(u)$, where $p\colon O(M)\to M$ is the projection map. We denote by $vX$ and $vY$ the vertical components of $X$ and $Y$, and by $\tilde{g}_x$ the Riemannian metric on the manifold $O_x(M)$, defined by the former construction. Then the metric $\bar{g}$ on $O(M)$ is defined by

$$\bar{g}(X,Y)=g(dp\,X,dpY)+\varrho\tilde{g}_x(vX,vY), \tag{2}$$

where $\varrho$ is an arbitrary positive constant.

## 3§. The Riemannian connection on $O(M)$

We denote by $\omega_i$ and $\omega_{ik}$ the components of the $R^n$-valued basic form and of the $o(n)$-valued Riemannian connection form on $O(M)$ respectively. It is well known that they define a parallelization of the total manifold $O(M)$. The Riemannian metric (1) can be expressed with help of these forms:

$$dp^*\,ds^2=\sum_{i=1}^{n}(\omega_i)^2+\varrho\sum_{i>k}(\omega_{ik})^2. \tag{3}$$

It follows that the parallelization defined by the coframe consisting from the forms

$$\Theta_i=\omega_i,\quad \Theta_{ik}=\sqrt{\varrho}\,\omega_{ik}$$

is orthonormal, and

$$dp^*\,ds^2=\sum_{i=1}^{n}(\Theta_i)^2+\sum_{i>k}(\Theta_{ik})^2.$$

**Theorem 1.** *The Riemannian connection on $O(M)$ can be reduced to the bundle of orthonormal absolute parallelization defined by the forms $\Theta_i$, $\Theta_{ij}$. The components $\Theta_{i,k}$, $\Theta_{ij,k}$. $\Theta_{i,kl}$, $\Theta_{ij,kl}$ of this connection form can be expressed as follows:*

$$\Theta_{i,k}=\omega_{ik}-\sqrt{\varrho}/2\sum_{l>m}R_{lmik}\Theta_{lm},$$

$$\Theta_{ij,k}=\sqrt{\varrho}/2R_{ijkm}\Theta_m,$$

$$\Theta_{i,kl}=-\sqrt{\varrho}/2R_{klim}\Theta_m,$$

$$\Theta_{ij,kl}=\frac{1}{2\sqrt{\varrho}}(\Theta_{ik}\delta_{jl}-\Theta_{il}\delta_{jk}+\Theta_{jk}\delta_{il}-\Theta_{lj}\delta_{ik}).$$

PROOF. Since the components of the metric tensor $\bar{g}$ are $\bar{g}_{ik}=\delta_{ik}$, $\bar{g}_{i,kl}=0$, $\bar{g}_{ij,kl}=\delta_{ik}\delta_{jl}$, it is sufficient to prove that the components $\Theta_{IK}$ are antisymmetric: $\Theta_{IK}+\Theta_{KI}=0$ and that the components of the torsion form $d\Theta_I+\Theta_{IK}\wedge\Theta_K$ vanish identically (the indices $I, K$ run over $1, \dots, n$ and over the pairs $(i, k)$, $i>k$). But they are easily verifiable consequences of the structure equations

$$d\omega_i = -\omega_{ik}\wedge\omega_k,$$
$$d\omega_{ik} = -\omega_{im}\wedge\omega_{mk}+\frac{1}{2}R_{iklm}\omega_l\wedge\omega_m.$$

## 4§. Geodesics on $O(M)$

In the following we denote by $s$ the arc-length parameter of curves on the basic manifold $M$ and by $\bar{s}$ the arc-length parameter of curves on $O(M)$. Let dash denote the derivation by $s$ and point the derivation by $\bar{s}$.

Let $u=(x, e_1, \dots, e_n)\in O(M)$, $X\in T_uO(M)$. We denote by $\lambda(X)$ the following element of $T_xM\wedge T_xM$:

$$\lambda(X) := \sum_{i>k}\omega_{ik}e_i\wedge e_k.$$

It is clear that the tensor of curvature $R$ at $x\in M$ can be regarded as a map $R_x\colon T_xM\wedge T_xM\otimes T_xM\to T_xM$, and so the expression $R(\lambda(X)\otimes dp(X))$ has meaning.

**Theorem 2.** *The curve $(x(s), e_1(s), \dots, e_n(s))$ is a geodesic on $O(M)$ with respect to the metric* (1) *if and only if*

a) *the first vector of curvature $\nabla_s x'$ of the curve $x(s)$ is*

$$\nabla_s x' = \varrho R(\lambda(u')\otimes x'),$$

*where $x'=dp(u')$,*

b) *the bivector field $\lambda(u')$ is parallel along $x(s)$,*

c) *the curve $\tau(s, s_0)e_1(s), \dots, \tau(s, s_0)e_n(s)$ is an affinely parametrized geodesic on $O_{x(s_0)}(M)$, where $\tau(s, s_0)\colon T_{x(s)}M\to T_{x(s_0)}M$ denotes the operator of parallel translation along $x(s)$.*

PROOF. Let $u(\bar{s})=(x(\bar{s}), e_1(\bar{s}), \dots, e_n(\bar{s}))$ be a curve on $O(M)$. It is a geodesic if and only if its tangent vector $u$ satisfies the differential equations

$$\frac{d}{d\bar{s}}\Theta_i+\Theta_{i,k}\Theta_k+\sum_{k>l}\Theta_{i,kl}\Theta_{kl} = 0,$$

$$\frac{d}{d\bar{s}}\Theta_{ij}+\Theta_{ij,k}\Theta_k+\sum_{k>l}\Theta_{ij,kl}\Theta_{kl} = 0.$$

Now, using Theorem 1 we can rewrite the above equations as follows:

$$\frac{d}{d\bar{s}}\omega_i+\omega_{ik}\omega_k-\varrho\sum_{k>l}R_{klim}\omega_{kl}\omega_m = 0,$$

$$\frac{d}{d\bar{s}}\omega_{ij} = 0,$$

where $\dot{x}=\omega_i(\dot{u})e_i$, $\nabla_{\bar{s}}e_i=\omega_{ik}(\dot{u})e_k$. These equations are in coordinate free formulation

(4) $$\nabla_{\bar{s}}\dot{x} = \varrho R(\lambda(\dot{u})\otimes\dot{x}),$$

(5) $$\nabla_{\bar{s}}(\lambda(\dot{u})) = 0.$$

The equation (5) means that the curve $(\tau(\bar{s}, \bar{s}_0)e_1, \dots, \tau(\bar{s}, \bar{s}_0)e_n)$ is an affinely parametrized geodesic on $O_{x(s0)}(M)$. Thus the length of its tangent vector is constant:

$$\tilde{g}(v\dot{u}, v\dot{u}) = \sum_{i>k}(\omega_{ik}(\dot{u}))^2 = \text{constant}.$$

From this follows that $g(dp(\dot{u}), dp(\dot{u}))=\sum_{i=1}^{n}(\omega_i(\dot{u}))^2=\text{constant}$, as well.

Let $b=\left(\sum_{i=1}^{n}(\omega_i(\dot{u}))^2\right)^{1/2}$.

If $b=0$, the conditions of Theorem are proved.

If $b>0$ we can rewrite the equation (3):

$$\nabla_s x' = \varrho R(\lambda(u')\otimes x'),$$

and so get the condition (a).

Now, let $(x(s), e_1(s), \dots, e_n(s))$ be a curve on $O(M)$ satisfying conditions $(a), (b), (c)$. Then

$$\nabla_s x' = R(\lambda(u')\otimes x'), \quad \nabla_s(\lambda(u')) = 0,$$

and so the equations of geodesics (4) and (5) are fulfilled.

## 5§. The locally symmetric case

Now we shall investigate the case, when the basic Riemannian manifold is locally symmetric, i.e. $\nabla R=0$.

Let a geodesic $(x(s), e_1(s), \dots, e_n(s))$ on $O(M)$ be given. Then by Theorem 2 $\nabla_s x'=\varrho R(\lambda(u')\otimes x')$. We denote the operator $R_x(\lambda(u'))$: $T_xM\to T_xM$ along the curve $x(s)$ by $K_x$ and the vector $x'$ by $v_1$. It is clear that the operator $K_x$ is antisymmetric with respect to the scalar product $g$. Moreover it follows from $\nabla R=0$ and from Theorem 2 that $\nabla_s K=0$.

The $i^{\text{th}}$ curvature $\varkappa_i$ and the $i^{\text{th}}$ vector of curvature $\varkappa_i v_{i+1}$ of the curve $x(s)$ on $M$ are defined by the generalized Frenet formulas

(6) $$\begin{cases}\nabla_s v_1 = \varkappa_1 v_2,\\ \nabla_s v_i = -\varkappa_{i-1}v_{i-1}+\varkappa_i v_{i+1},\\ \nabla_s v_n = -\varkappa_{n-1}v_{n-1},\end{cases}$$

$i=2, \dots, n-1.$

**Lemma.** *The curvatures $\varkappa_1, \dots, \varkappa_{n-1}$ of the curve $x(s)$ are constant.*

PROOF. We shall prove the Lemma by induction on $i$. The vectorfields $w_1, \dots, w_n$ along $x(s)$ are defined by the recurrence

$$w_1 = v_1, \quad w_i = \nabla_s w_{i-1} = \varrho^{i-1}K^{i-1}v_1.$$

We assume $\varkappa_1, \dots, \varkappa_{i-1}$ are constant. Then the vectors of the Frenet frame can be expressed as linear combinations $v_i = \sum_{j=1}^{i} c_j w_j$, where the coefficients $c_1, \dots, c_i$ are constant.

It follows that $\nabla_s v_i = \sum_{j=1}^{i} c_j w_{j+1}$.

Now, we get that $g(\nabla_s v_i, \nabla_s v_i)$ is constant. In fact,

$$\nabla_s\big(g(\nabla_s v_i, \nabla_s v_i)\big) = 2\varrho g\left(K \sum_{j=1}^{i} c_j w_{j+1}, \sum_{j=1}^{i} c_j w_{j+1}\right) = 0$$

since $K$ is antisymmetric. But we have by (5)

$$g(\nabla_s v_i, \nabla_s v_i) = \varkappa_{i-1}^2 + \varkappa_i^2,$$

thus $\varkappa_{i-1}$=constant implies that $\varkappa_i$=constant as well.

**Theorem 3.** *Let $(x(s), e_1(s), \dots, e_n(s))$ be a geodesic on $O(M)$. Then the $i^{\text{th}}$ vector of curvature of the curve $x(s)$ can be expressed in the form*

$$\varkappa_i v_{i+1} = (\varkappa_1 \cdot \ldots \cdot \varkappa_{i-1})^{-1}(\mu_0^{(i)} w_{i+1} + \mu_1^{(i)} w_{i-1} + \ldots + \mu_m^{(i)} w_{i+1-2m}),$$

$(i+1-2m \geqq 1)$, *where* $\mu_j^{(k)} = p_{k,j}(\varkappa_1, \dots, \varkappa_{k-1})$ *and the polynomials* $p_{k,j}(x_1, \dots, x_{k-1})$ $(k \geqq 0)$ *are defined by the recurrence*

$$p_{k,0} = 1,$$

$$p_{k,j} = p_{k-1,j} + x_{k-1}^2 p_{k-2,j-1}, \quad \textit{if} \quad k+1 > 2j,$$

*and* $$p_{k,j} = 0 \qquad \textit{if} \quad k+1 \leqq 2j,\ j \neq 0.$$

The proof is the same as of Theorem 3 in [3].

## 6 §. On the orthonormal frame bundle of the $n$-sphere

We denote by $S_r^n$ the $n$-sphere of radius $r$. It is well known that there is a natural diffeomorphism $\varphi$ of $O(S_r^n)$ on the manifold $V^{n+1}$ of orthonormal frames in $R^{n+1}$, which maps an $n$-frame $(x, e_1, \dots, e_n) \in O(S_r^n)$ on the $n+1$-frame $(\frac{1}{r} x, e_1, \dots, e_n) \in V^{n+1}$.

**Theorem 4.** *The map $\varphi$: $O(S_r^n) \to V^{n+1}$ is a homothety with respect to the Riemannian metrics* (2) *and* (1), *respectively, if and only if $r^2 = \varrho$. In this case the ratio of homothety is $r^2$.*

PROOF. Let $u = (x, e_1, \dots, e_n) \in O(S_r^n)$ and $\varphi(u) = (e_0, e_1, \dots, e_n) \in V^{n+1}$, where $e_0 = \frac{1}{r} x$. We can write

$$dx = \omega_i e_i, \quad de_i = -\omega_i x + \omega_{ik} e_k,$$

and

$$de_0 = \omega_{0i} e_i, \quad de_i = \omega_{i0} e_0 + \omega_{ik} e_k,$$

respectively, where $i, k=1, \dots, n$. From these follows $\omega_i = r\omega_{oi}$, since $x = re_0$. The Riemannian metrics on $O(S_r^n)$ and on $V^{n+1}$ are

$$d\bar{s}^2 = \sum_i (\omega_i)^2 + \varrho \sum_{i>k} (\omega_{ik})^2 = r^2 \sum_i (\omega_{i0})^2 + \varrho \sum_{i>k} (\omega_{ik})^2$$

and

$$d\tilde{s}^2 = \sum_i (\omega_{i0})^2 + \sum_{i>k} (\omega_{ik})^2.$$

respectively, from which the Theorem follows.

## References

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