# Remarks on the limit inferior of a filtered set-family

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It is well-known (see KURATOWSKI [6], p. 242) that in each metric space $E$ the limit inferior $\liminf_{n\to\infty} A_n$ of each sequence $(A_n)$ with $A\subseteqq E$ is the set of all $x\in E$ such that there exist an $n_0(x)\in\mathbf{N}$ and a sequence $(a_n(x))$ in $E$ with $a_n(x)\in A_n$ for all $n\geqq n_0(x)$ and $x=\lim_{n\to\infty} a_n(x)$. In this paper, we give (in Proposition 3) a modified generalization of this statement to a (general) topological space $E$ and the limit inferior $\liminf(f, I, \mathfrak{a})$ of a filtered family $(f, I, \mathfrak{a})$ with $f(i)\subseteqq E$ for all $i\in I$.

For the remainder, let $(E, \tau)$ be a topological space, $\mathfrak{B}_\tau$ the neighborhood operator, $\mathrm{Lim}_\tau$ the limit operator, $\liminf_\tau$ the limit inferior induced by the topology $\tau$. For abbreviation, we write just $\mathfrak{B}$, Lim, lim inf instead of $\mathfrak{B}_\tau$, $\mathrm{Lim}_\tau$, $\liminf_\tau$ if no confusion can arise.

**1. Terminology.** In every respect we shall use the same terminology as used or introduced in [4]. For nonempty sets $I$ and $K$ and filters $\mathfrak{a}$ and $\mathfrak{b}$ on $I$ and $K$, respectively, $\mathfrak{a}\otimes\mathfrak{b}$ denotes the ordinal product of $\mathfrak{a}$ and $\mathfrak{b}$ (which is a filter on $I\times K$) (see [3], p. 330 and p. 336, Satz 23). Given a filter $\mathfrak{a}$ on a set $I$ and $A\in\mathfrak{a}$, $\mathfrak{a}_A$ denotes the trace of $\mathfrak{a}$ in the set $A$; furthermore, if, for each $i\in I$, a statement form $H(i)$ containing $i$ as a free variable is given, we say that $H(i)$ holds "for $\mathfrak{a}$-almost all $i\in I$" if and only if, for some $A\in\mathfrak{a}$, $H(i)$ holds for all $i\in A$. Given a mapping $f$ and a set $B\subseteqq\mathscr{D}f$, we denote the restriction of $f$ to $B$ by $f_B$. Recall that, for each set $M$, $\Phi M$ denotes the class of all filtered families in $M$. For each directed set $(D, \leqq)$, i.e. for each set $D$ with a reflexive and transitive relation $\leqq$ such that each finite subset of $D$ has an upper bound w.r. to $\leqq$, we denote by $\mathfrak{F}D$ the "filter of perfinality" on $D$, which is defined to be the filter on $D$ generated by the set $\{\{z\mid y\leqq z\in D\}\mid y\in D\}$. Especially, for each $x\in E$, $(\mathfrak{B}x, \supseteqq)$ is a directed set, and so $\mathfrak{F}(\mathfrak{B}x)$ is the filter of perfinality on $\mathfrak{B}x$.

**2. Connection between** lim inf **and** Lim. The mapping lim inf is an extension of the mapping Lim in the sense of

**Proposition 1.** *Let $\varkappa$ denote the mapping $x\mapsto\{x\}$ on $E$ into $\mathfrak{P}E$. Then*

$$\mathrm{Lim}\,(f, I, \mathfrak{a}) = \liminf(\varkappa\circ f, I, \mathfrak{a})$$

*holds for all $(f, I, \mathfrak{a})\in\Phi E$.*

PROOF. Use of the definitions only. □

While Proposition 1 remains true in general finitely additive quasitopological spaces (see Section 4), this is not the case for the next proposition:

**Proposition 1′.** *In the notation of Proposition 1, one has:*

$$\operatorname{Lim}(f, I, \mathfrak{a}) = \liminf(\tau \circ \varkappa \circ f, I, \mathfrak{a})$$

*holds for all* $(f, I, \mathfrak{a}) \in \Phi E$.

PROOF. Use that, for each $x \in E$, $\mathfrak{B}x$ is generated by the class of all open neighborhoods of $x$; furthermore that $f(i) \in \tau\{f(i)\}$ for all $i \in I$. □

One half of the statement on $\liminf_{n\to\infty} A_n$ in the introduction is still true in general topological spaces:

**Proposition 2.** *For all* $x \in E$ *and all* $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$, *the following statement form* (a) *implies* (b):

(a) *There are an* $A \in \mathfrak{a}$ *and a* $g \in \mathop{\mathbf{P}}_{i \in A} f(i)$ *such that* $x \in \operatorname{Lim}(g, A, \mathfrak{a}_A)$.
(b) $x \in \liminf(f, I, \mathfrak{a})$.

PROOF. Use of the definitions of Lim and lim inf in terms of $\mathfrak{B}$. □

The full generalization of the introductory remark on $\liminf_{n\to\infty} A_n$ is given by:

**Proposition 3.** *For all* $x \in E$ *and all* $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$, *the following statement forms* (a) *through* (d) *are equivalent:*

(a) $x \in \liminf(f, I, \mathfrak{a})$;
(b) *There exists a mapping* $g$ *on* $(\mathfrak{B}x) \times I$ *into* $E$ *with the following property:* $g(V, i) \in f(i)$ *for* $((\mathfrak{F}(\mathfrak{B}x)) \otimes \mathfrak{a})$*-almost all* $(V, i) \in (\mathfrak{B}x) \times I$ *and* $x \in \operatorname{Lim}\big(g, (\mathfrak{B}x) \times I, (\mathfrak{F}(\mathfrak{B}x)) \otimes \mathfrak{a}\big)$.
(c) *is obtained from* (b) *by replacing the quantifier "for* $((\mathfrak{F}(\mathfrak{B}x)) \otimes \mathfrak{a}$*-almost all" by "for* $\{\mathfrak{B}x\} \otimes \mathfrak{a}$*-almost all".*
(d) *There are a* $C \in (\mathfrak{F}(\mathfrak{B}x)) \otimes \mathfrak{a}$ *and a* $g \in \mathop{\mathbf{P}}_{(V, i) \in C} f(i)$ *such that*

$$x \in \operatorname{Lim}\big(g, C, ((\mathfrak{F}(\mathfrak{B}x)) \otimes \mathfrak{a})_C\big).$$

PROOF. First recall that

$$(1) \qquad \operatorname{Lim}(h, K, \mathfrak{x}) = \operatorname{Lim}(h_X, X, \mathfrak{x}_X)$$

holds for each $(h, K, \mathfrak{x}) \in \Phi E$ and each $X \in \mathfrak{x}$. Let $x \in E$ and $(f, I, \mathfrak{a}) \in \Phi(\mathfrak{P}E)$. For abbreviation we put $\mathfrak{B}x = D$. Now we subdivide the proof.

*1.* By (1), (b) and (d) are equivalent.
*2.* Since $D \in \mathfrak{F}D$, (c) implies (b).
*3.* We show that (a) implies (c). Suppose (a). Then, there is, for each $V \in D$, a set $A_V \in \mathfrak{a}$ such that

$$(2) \qquad V \cap f(i) \neq \emptyset \quad \text{for all} \quad i \in A_V.$$

Let $(A_V)_{V \in D}$ be a family of such sets $A_V$ (use of the axiom of choice) and $\mathfrak{B}$ the direct sum $\mathop{\mathbf{S}}_{V \in D} A_V$ of the sets $A_V$ (see [3], p. 325). Then (by (2)) there exists a $\chi \in \mathop{\mathbf{P}}_{(V, i) \in \mathfrak{B}} V \cap f(i)$ (use of the axiom of choice). Define a mapping $g$ on $D \times I$ into $E$ by letting

$$g(V, i) = \begin{cases} \chi(V, i) & \text{for all} \quad (V, i) \in \mathfrak{B}, \\ x & \text{for all} \quad (V, i) \in (D \times I) \setminus \mathfrak{B}. \end{cases}$$

Since $\mathfrak{B}\in\{D\}\otimes\mathfrak{a}$, we have

$$g(V, i)\in f(i) \quad \text{for} \quad \{D\}\otimes\mathfrak{a}\text{-almost all} \quad (V, i)\in D\times I.$$

Let $W\in D$. Then, by the definition of $g$,

$$g(V, i)\in V\cap f(i)\subseteqq W\cap f(i)\subseteqq W$$

holds for all $(V, i)$ with $V\subseteqq W$, $V\in D$ and $i\in A_V$, thus for $(\mathfrak{F}D)\otimes\mathfrak{a}$-almost all $(V, i)\in D\times I$. By the choice of $W$, we obtain $x\in\mathrm{Lim}\,(g, D\times I, (\mathfrak{F}D)\otimes\mathfrak{a})$. Therefore, (c) holds.

*4.* We show that (b) implies (a). Suppose (b). Let $V\in D$. Then, with the mapping $g$ taken from (b), $g(W, i)\in V\cap f(i)$ holds for $(\mathfrak{F}D)\otimes\mathfrak{a}$-almost all $(W, i)\in D\times I$ (consider that $(\mathfrak{F}D)\otimes\mathfrak{a}$ is a filter). Therefore, by the definition of $\mathfrak{F}D$ and of $\otimes$ (use of [3], p. 330, Satz 13), there exists a set $U\in D$ and, for each $W\in D$ with $W\subseteqq U$, a set $A_W\in\mathfrak{a}$ such that

$$g(W, i)\in V\cap f(i) \quad \text{for all} \quad i\in A_W.$$

Especially, we have $V\cap f(i)\neq\emptyset$ for all $i\in A_U$. By the choice of $V$, we obtain (a). □

**3. Applications of the preceding propositions.** It would be desirable to regain "nice" properties of certain subspaces of $(\mathfrak{P}E, \mathfrak{P}\tau)$ (the power of $(E, \tau)$) from corresponding or related properties of $(E, \tau)$, or conversely, by means of the preceding propositions. A simple step in the desired direction can be seen in the next proposition.

**Proposition 4**. *Let $\mathfrak{M}\subseteqq(\mathfrak{P}E)\setminus\{\emptyset\}$ and assume $\{x\}\in\mathfrak{M}$ for all $x\in E$. Then, the subspace $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ of $(\mathfrak{P}E, \mathfrak{P}\tau)$ is compact if and only if $(E, \tau)$ is compact.*

Proof. *1.* Assume $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ to be compact. Let $(f, I, \mathfrak{a})\in\Phi E$ with an ultrafilter $\mathfrak{a}$. In the terminology within Proposition 1, one has $\mathfrak{M}\cap(\mathfrak{P}\,\liminf\,(\varkappa\circ f, I, \mathfrak{a}))\neq\emptyset$, since $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ is compact. Therefore, there is an $M\in\mathfrak{M}$ with $M\subseteqq\liminf\,(\varkappa\circ f, I, \mathfrak{a})=\mathrm{Lim}\,(f, I, \mathfrak{a})$ (use of Proposition 1), thus (since $M\neq\emptyset$) Lim $(f, I, \mathfrak{a})\neq\emptyset$. Therefore, by the choice of $(f, I, \mathfrak{a})$, $(E, \tau)$ is compact.

*2.* Assume that $(E, \tau)$ is compact. Let $(f, I, \mathfrak{a})\in\Phi\mathfrak{M}$ with an ultrafilter $\mathfrak{a}$. Since $\emptyset\notin\mathfrak{M}$, there is a $g\in\mathop{\mathbf{P}}_{i\in I} f(i)$ (axiom of choice); then, $(g, I, \mathfrak{a})\in\Phi E$ and $\mathrm{Lim}\,(g, I, \mathfrak{a})$ is nonempty, since $(E, \tau)$ is compact. In view of Proposition 2, $\liminf\,(f, I, \mathfrak{a})$ is nonempty; choose $x\in\liminf\,(f, I, \mathfrak{a})$. Then $\{x\}\in\mathfrak{M}\cap(\mathfrak{P}\,\liminf\,(f, I, \mathfrak{a}))$. By the choice of $(f, I, \mathfrak{a})$, $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ is proved to be compact. □

While the proof of Proposition 4 can be carried over, word by word, to finitely additive quasitopological spaces (see Section 4), this is not the case for the proof of the next proposition:

**Proposition 4′**. *Let $\mathfrak{M}\subseteqq(\mathfrak{P}E)\setminus\{\emptyset\}$ and assume $\tau\{x\}\in\mathfrak{M}$ for all $x\in E$. Then, the subspace $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ of $(\mathfrak{P}E, \mathfrak{P}\tau)$ is compact if and only if $(E, \tau)$ is compact.*

Proof. *1.* In Part 1 of the preceding proof, replace $(\varkappa\circ f, I, \mathfrak{a})$ by $(\tau\circ\varkappa\circ f, I, \mathfrak{a})$ and the reference to Proposition 1 by that to Proposition 1′.

*2.* In Part 2 of the preceding proof, replace $\{x\}$ by $\tau\,\{x\}$. Use that $\liminf\,(f, I, \mathfrak{a})$ is closed there. □

*Remark*. Proposition 4 contains the special case $\mathfrak{M}=\mathfrak{P}E\setminus\{\emptyset\}$. In this case, $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ coincides (see [5]) with the hyperspace of lower semicontinuity of $(E, \tau)$ (see MICHAEL [7], p. 179, Definition 9.1 ("$\mathfrak{P}E\setminus\{\emptyset\}$ with the lower finite topology"), and (for "closure spaces") ČECH [1], p. 623, Definition 34 A.1). Proposition 4′ contains the special case $\mathfrak{M}=2^E$ (= set of all nonempty closed subsets of $(E, \tau)$). In this case, $(\mathfrak{M}, (\mathfrak{P}\tau)_{\mathfrak{M}})$ coincides (see [5] or FLACHSMEYER [2], p. 326, 2.1, or POPPE [8]) with Michael's space $2^E$ endowed with the "lower finite topology" ([7], loc. cit.). While this special case of Proposition 4′ occurs in the literature (with a different proof, using Alexander's Lemma (see FLACHSMEYER [2], p. 327, 2.4)), the author could not find a reference concerning the mentioned special case of Proposition 4.

In the paper [4], we have considered topological spaces $(E, \tau)$, $(F, \sigma)$, $(G, \lambda)$, relations $R\subseteq E\times F$, $S\subseteq F\times G$ with $\mathscr{R}R\subseteq\mathscr{D}S$, and — respectively — the canonical mapping $\hat{R}$, $\hat{S}$, $\widehat{S\circ R}$ induced by $R$, $S$, $S\circ R$, which is a mapping *from* (in general not *on*!) $(E, \tau)$, $(F, \sigma)$, $(E, \tau)$ into the power $(\mathfrak{P}F, \mathfrak{P}\sigma)$, $(\mathfrak{P}G, \mathfrak{P}\lambda)$, $(\mathfrak{P}G, \mathfrak{P}\lambda)$ of $(F, \sigma)$, $(G, \lambda)$, $(G, \lambda)$, as we do now for the following. We recall the validity of the logical diagram

$$\begin{array}{ccccc} & R \text{ and } S \text{ continuous} & \Rightarrow & \hat{R} \text{ and } \hat{S} \text{ continuous} & \\ \Downarrow & & & & \Downarrow \\ & S\circ R \quad \text{continuous} & \Rightarrow & \widehat{S\circ R} \text{ continuous} & \end{array}$$

and use Proposition 3 to reprove the right-hand arrow, more precisely, to reprove the next proposition (cf. ČECH [1], p. 631, Theorem 34 B. 14, and [5]; furthermore, see [4], p. 41, Proposition 7):

**Proposition 5**. *If $\hat{R}$ is $(\tau, \mathfrak{P}\sigma)$-continuous and $\hat{S}$ is $(\sigma, \mathfrak{P}\lambda)$-continuous, then $\widehat{S\circ R}$ is $(\tau, \mathfrak{P}\lambda)$-continuous.*

PROOF. Let $(f, I, \mathfrak{a})\in\Phi(\mathscr{D}R)$ and $x\in(\mathscr{D}R)\cap\operatorname{Lim}_\tau(f, I, \mathfrak{a})$. Let $y\in(\widehat{S\circ R})(x)$ and $U\in\mathfrak{B}_\lambda y$. Then, there exists a $z\in F$ with $(x, z)\in R$ and $(z, y)\in S$. Since $\hat{R}$ is $(\tau, \mathfrak{P}\sigma)$-continuous, we have $\hat{R}(x)\subseteq\liminf_\sigma\,(\hat{R}\circ f, I, \mathfrak{a})$. Therefore, since $z\in\hat{R}(x)$, $z\in\liminf_\sigma(\hat{R}\circ f, I, \mathfrak{a})$. For abbreviation, we set $(\mathfrak{F}(\mathfrak{B}_\sigma z))\otimes\mathfrak{a}=\mathfrak{c}$. By Proposition 3, there is a set $C\in\mathfrak{c}$ and a mapping $g\in\underset{(V,i)\in C}{\mathbf{P}}(\hat{R}\circ f)(i)$ such that

$$z\in\operatorname{Lim}_\sigma(g, C, \mathfrak{c}_C). \tag{3}$$

Since $\mathscr{R}R\subseteq\mathscr{D}S$ (by supposition), one has $\mathscr{D}(\hat{S}\circ g)=C$ (in view of the choice of $g$). Because $\hat{S}$ is $(\sigma, \mathfrak{P}\lambda)$-continuous, one obtains by (3)

$$\hat{S}(z)\subseteq\liminf_\lambda(\hat{S}\circ g, C, \mathfrak{c}_C). \tag{4}$$

Since $y\in\hat{S}(z)$ and $U\in\mathfrak{B}_\lambda y$, there exists, by (4), a set $D\in\mathfrak{c}_C$ such that

$$U\cap((\hat{S}\circ g)(V, i))\neq\emptyset \quad \text{for all} \quad (V, i)\in D. \tag{5}$$

Because $\mathfrak{c}_C\subseteq\mathfrak{c}$, there are (by "Satz 13" in [3], p. 330) a set $W\in\mathfrak{B}_\sigma z$ and a family $(A_V)_{V\in\mathfrak{B}}$, where $\mathfrak{B}$ denotes the set of all $V\in\mathfrak{B}_\sigma z$ with $V\subseteq W$, such that $A_V\in\mathfrak{a}$ for all $V\in\mathfrak{B}$ and $\underset{V\in\mathfrak{B}}{\mathbf{S}}A_V\subseteq D$. We set $A_W=J$ (for abbreviation) and choose a mapping $\chi\in\underset{i\in J}{\mathbf{P}}(U\cap((\hat{S}\circ g)(W, i)))$ (use of (5) and the axiom of choice).

Let $i \in J$. Then, by the choice of $g$, $(f(i), g(W, i)) \in R$ and, by the choice of $\chi$, $(g(W, i), \chi(i)) \in S$, thus $(f(i), \chi(i)) \in S \circ R$, therefore $\chi(i) \in (\widehat{S \circ R})(f(i))$. On the other hand, we have $\chi(i) \in U$, thus, for all $i \in J$ (therefore, for $\mathfrak{a}$-almost all $i \in I$), $U \cap ((\widehat{S \circ R})(f(i))) \neq \emptyset$. In view of the choice of $U$, we obtain

$$y \in \liminf_{\lambda} ((\widehat{S \circ R}) \circ f, I, \mathfrak{a}).$$

We have proved that $\widehat{S \circ R}$ is $(\tau, \mathfrak{P}\lambda)$-continuous. □

**4. Generalization**. The Propositions 1, 2, 3, 4, 5 and their proofs remain valid if the topological spaces $(E, \tau)$, $(F, \sigma)$, $(G, \lambda)$ are replaced by general finitely additive quasitopological spaces (terminology: [5]) except for the following change within Proposition 3: Within (b), replace "There exists" by the words "$\mathfrak{V}x$ is a filter and there exists". Within (d), replace "There are" by the words "$\mathfrak{V}x$ is a filter and there are". — A quasitopological space $(E, \tau)$ is called to be compact if and only if $(\mathrm{Lim}_{\tau}\, \mathfrak{a} \neq \emptyset$ for all ultrafilters $\mathfrak{a}$ in $E$.

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