Concrete radicals in general modules

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§ 1. Introduction

Let A be an associative ring, (without any assumptions on commutativity or the existence of a unity element), and denote by ${}_A\mathcal{M}$ the category of all left A-modules. For this category we take over the concept of a radical as it is employed in *torsion theory* for unitary modules over a ring with unity element; (see e.g. [6]). We define: *A preradical* of ${}_A\mathcal{M}$ is a functor $r\colon {}_A\mathcal{M} \to {}_A\mathcal{M}$ such that for all $M, N \in {}_A\mathcal{M}$,

(i) $M \mapsto r(M)$, a submodule of M;
(ii) $(f \in \mathrm{Hom}_A(M, N)) \mapsto r(f) = f|_{r(M)} \in \mathrm{Hom}_A(r(M), r(N))$.

A preradical r is said to be *idempotent* if $r(r(M)) = r(M)$ for all $M \in {}_A\mathcal{M}$, and r is called a *radical* if $r(M/r(M)) = 0$ for all $M \in {}_A\mathcal{M}$. If r is a radical the submodule $r(M)$ is called the *r-radical* of the module M, and M is said to be *r-radical (r-semisimple)*, if $r(M) = M$, $(r(M) = 0)$.

Our purpose is to construct a class of radicals of this type and to produce a few concrete examples of such radicals.

§ 2. Radicals of ${}_A\mathcal{M}$ defined by intersections

The basic tool in our construction is the well known one used in the theory of radicals in rings, namely *taking intersections*. Some terminology will be necessary: We consider a property Σ of submodules which defines within each module $M \in {}_A\mathcal{M}$ a subsystem Σ_M of submodules, called the *Σ-submodules* of M. For a given Σ the system Σ_M may be empty, e.g. when the Σ-submodules are defined to be the maximal submodules. A defining property as described above will be called an *isolator*.

2.1 Definition. An isolator Σ is said to be *stable* on ${}_A\mathcal{M}$ if for each A-epimorphism $f\colon M \to N$ the assignment $P \mapsto f(P)$ defines a bijection between $\{P \in \Sigma_M \mid \mathrm{Ker}\, f \subset P\}$ and Σ_N.

A direct consequence of this definition is that $S \in \Sigma_M$ if and only if $0 \in \Sigma_{M/S}$.

2.2 Definition. A *transferring* isolator Σ is one which satisfies the following condition: If $M \in {}_A\mathcal{M}$, T a submodule of M and $S \in \Sigma_M$, then $T \cap S \neq T$ implies that $T \cap S \in \Sigma_T$.

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With each stable transferring isolator Σ we now associate the assignment $r\colon {}_A\mathcal{M}\to {}_A\mathcal{M};\ M\mapsto r(M)$ and $(f\colon M\to N)\mapsto r(f)=f|_{r(M)}\colon r(M)\to N$; where we define $r(M)=\bigcap_{S\in\Sigma_M} S$ if $\Sigma_M\neq\emptyset$ and $r(M)=M$ if $\Sigma_M=\emptyset$. We show that this assignment defines a radical of ${}_A\mathcal{M}$.

2.3 Lemma. *If $M\in {}_A\mathcal{M}$ and T is a submodule of M, then $r(T)\subset T\cap r(M)$.*

PROOF. If either $\Sigma_M=\emptyset$ or $T\subset S$ for all $S\in\Sigma_M$, there is nothing to prove. If $T\cap S\neq T$ for at least one $S\in\Sigma_M$, then $T\cap S\in\Sigma_T$ and hence $r(T)= =\bigcap_{K\in\Sigma_T} K\subset \bigcap_{S\in\Sigma_M}(T\cap S)=T\cap\big(\bigcap_{S\in\Sigma_M} S\big)=T\cap r(M)$.

2.4 Theorem. *r is a radical of ${}_A\mathcal{M}$, i.e. every A-homomorphism $f\colon M\to N$ induces $r(M)\to r(N)$ by restriction and $r\big(M/r(M)\big)=0$ for all $M\in {}_A\mathcal{M}$.*

PROOF. To prove the first part of the theorem we consider an arbitrary A-homomorphism $f\colon M\to N$. We need obviously only consider the case where $r(N)\neq N$. If $r(M)=M$, then the stableness of Σ shows that $r(\mathrm{Imf})=\mathrm{Imf}$. Since now $\mathrm{Imf}\subset r(N)$, by 2.3, we have that $r(M)\to r(N)$. If $r(M)\neq M$ we consider the set $\{S_\alpha\in\Sigma_M \mid \operatorname{Ker} f\subset S_\alpha\}$, which is non-empty since $r(N)\neq N$. Now

$$f\big(r(M)\big)\subset f\big(\bigcap_\alpha S_\alpha\big)\subset \bigcap_\alpha f(S_\alpha) = r(\mathrm{Imf})\subset \mathrm{Imf}\cap r(N)\subset r(N).$$

Hence $r(M)\to r(N)$ in this case also.

Concerning the second part of the theorem we may clearly confine ourselves to modules $M\in {}_A\mathcal{M}$ with $0\neq r(M)\neq M$. Now considering the canonical projection $M\to M/r(M)$ we have the bijection $\Sigma_M\leftrightarrow\Sigma_{M/r(M)}$, $K_\beta\leftrightarrow K_\beta/r(M)$. So if $x+r(M)\in r\big(M/r(M)\big)$ then $x+r(M)\in K_\beta/r(M)$ for all β so that $x\in K_\beta$ for all β. Hence $x\in r(M)$, or equivalently, $x+r(M)=0$. This completes the proof.

As an immediate consequence of the construction of $r(M)$ we also have the following criterion.

2.5 Corollary. *A non-zero module M in ${}_A\mathcal{M}$ is isomorphic to a subdirect product of all modules M/S, $(S\in\Sigma_M)$, if and only if $r(M)=0$.*

§ 3. Concrete radicals in ${}_A\mathcal{M}$

In view of theorem 2.4 we may now substitute different stable transferring isolators for Σ to obtain concrete examples of radicals of ${}_A\mathcal{M}$. Our basic isolator (which we shall denote by Σ^1) is defined as follows:

3.1 Definition. A submodule S of a module $M\in {}_A\mathcal{M}$ shall be called a *Σ^1-submodule* of M if $x\in M$ and $Ax\subset S$ imply that $x\in S$.

3.2 Lemma. *Σ^1 is a stable transferring isolator.*

PROOF. Let $f\colon M\to N$ be an A-epimorphism and let $P\in\Sigma^1_M$ such that $\operatorname{Ker} f\subset P$. Let $An\subset f(P)$ and m a fixed element of M with $f(m)=n$. Then $Af(m)\subset f(P)$ implies that $f(am)\in f(P)$ for all $a\in A$. Hence for each $a\in A$ there is a $p_a\in P$ such that $f(am-p_a)=0$ so that $am-p_a\in\operatorname{Ker} f\subset P$. This shows that $Am\subset P$. Since

$P\in\Sigma^1_M$ we must have that $m\in P$ and consequently $n\in f(P)$. Hence $f(P)\in\Sigma^1_N$. On the other hand, if $Q\in\Sigma^1_N$, then $Am_1\subset f^{-1}(Q)$ implies that $Af(m_1)\subset Q$ so that $f(m_1)\in Q$ and hence $m_1\in f^{-1}(Q)$. Thus we have that $f^{-1}(Q)\in\Sigma^1_M$. From the identity $f(f^{-1}(Q))=Q$ for all $Q\in\Sigma^1_N$ we obtain the surjectivity of the mapping $\{P\in\Sigma^1_M|\mathrm{Ker}\, f\subset P\}\to\Sigma^1_N$, $P\mapsto f(P)$. The injectivity follows from equally well known elementary arguments. Thus the stableness of Σ^1 is established. That this isolator also has the transferring property is a direct consequence of its definition.

The following properties of Σ^1 relative to an arbitrary $M\in{}_A\mathscr{M}$ are stated for reference; the verifications are straightforward.

3.3 Lemma. (i) *Σ^1_M is closed under intersections.* (ii) *If P, Q and R are submodules of M such that $P\in\Sigma^1_Q$ and $Q\in\Sigma^1_R$, then $P\in\Sigma^1_R$.* (iii) *If S is a submodule of M and $\bar{S}=\{x\in M\,|\,Ax\subset S\}$, then $S\in\Sigma^1_M$ if and only if $S=\bar{S}$.*

We now consider the radical r_1 associated with Σ^1. Let $M\in{}_A\mathscr{M}$. *Then* 3.3 (i) *shows that $r_1(M)=0$ if and only if $0\in\Sigma^1_M$*. Consequently, the subdirect decomposition of an r_1-semisimple module according to 2.5 is a trivial representation. Next we observe that r_1 is idempotent. For if $r_1(r_1(M))\subsetneqq r_1(M)$ for some $M\in{}_A\mathscr{M}$, then there is a $Q\subsetneqq r_1(M)$ in $\Sigma^1_{r_1(M)}$. Since $r_1(M)\in\Sigma^1_M$ by 3.3 (i), the property 3.3 (ii) shows that $Q\in\Sigma^1_M$. This contradicts the definition of $r_1(M)$.

Finally in connection with r_1 we characterize $r_1(M)$ under various conditions, mainly on the ground ring A. First we note that the submodules

$$M_n=\{x\in M\,|A^n x=0\},$$

$n\in\mathbf{N}$, are all contained in $r_1(M)$, for $x\in M_n$ implies that $A^n x=0$ so that $A^n x\subset r_1(M)$. This shows that $A(a'x)\subset r_1(M)$ for all $a'\in A^{n-1}$ so that $A^{n-1}x\subset r_1(M)$, because $r_1(M)\in\Sigma^1_M$. Continuing this process, we eventually obtain $Ax\subset r_1(M)$ and then $x\in r_1(M)$. Since clearly $M_n\subset M_{n+1}$ for all $n\in\mathbf{N}$ we have an ascending chain

$$M_1\subset M_2\ldots\subset\ldots\subset M_n\subset\ldots\subset r_1(M), \tag{1}$$

in which $M_1=\{x\in M\,|\,Ax=0\}$ is the maximal trivial submodule of M.

3.4 Corollary. *If there is an $n\in\mathbf{N}$ with $A^n=A^{n+1}=\ldots$, then $r_1(M)=M_n$.*

Proof. If $Ax\subset M_n$ then $A^n(ax)=0$ for all $a\in A$. Hence $A^{n+1}x=0$ so that $A^n x=0$, or equivalently $x\in M_n$. Thus we have that $M_n\in\Sigma^1_M$ which together with $M_n\subset r_1(M)$ show that $r_1(M)=M_n$. In particular, if A is left or right artinian there exists an $n\in\mathbf{N}$ such that $r_1(M)=M_n$, and if A is idempotent then $r_1(M)=M_1$. Apart from conditions on A we note that the chain (1) may also be employed in the same direct manner as in 3.4 to derive the following characterization.

3.5 Corollary. *If M is noetherian, then $r_1(M)=M_n$ for some $n\in\mathbf{N}$.*

Returning to our arbitrary $M\in{}_A\mathscr{M}$ we assume for the moment that A has a unity element e. Then 3.4 shows that $r_1(M)=M_1$. Furthermore, if $\lambda\colon A\to\mathrm{End}^l(M)$ is the ring homomorphism which supplies the module structure of M and if we identify $\lambda(e)$ with e, we note that the submodule $E=\{ex-x|x\in M\}$ is contained in $M_1=r_1(M)$, because $a(ex-x)=0$ for all $a\in A$. However, $E\in\Sigma^1_M$, for $Ay\subset E$ implies that $ey\in E$ so that $y\in E$. Hence we have that $r_1(M)=M_1=E$. Using the characterization $r_1(M)=M_1$ we see that r_1-radicality is equivalent with triviality, while the characterization $r_1(M)=E$ establishes the equivalence between r_1-semisimplicity and unitariness.

Once again we let A be an arbitrary ring and we now consider the Σ^1-*maximal* submodules of an arbitrary module $M\in{}_A\mathcal{M}$; these submodules will be termed the Σ^2-*submodules* of M. It is obvious that $\Sigma^2\subset\Sigma^1$, and the restriction of $P\mapsto f(P)$ in the proof of 3.2 to $\{P\in\Sigma^2_M|\text{Ker}\, f\subset P\}$ immediately yields the stableness of Σ^2. Moreover, if T is a submodule of M and $L\in\Sigma^2_M$, a direct application of the definition of Σ^1 shows that $T\cap L\neq T$ implies that $T\cap L\in\Sigma^1_T$. The relation $T\cap L\in\Sigma^2_T$ now follows from $T/(T\cap L)\cong(T+L)/L=M/L$, which is a simple module. Hence we have shown that Σ^2 is a stable transferring isolator, and we may consider the radical r_2 associated with it. First we observe that for every unitary Z-module (abelian group) M, $r_2(M)=\Phi(M)$, the Frattini submodule of M, and hence that r_2 is not idempotent. The latter observation immediately yields the expected fact that $r_1\neq r_2$. We shall employ the radical r_2 to fit into our scheme a well known module radical:

3.6 Theorem. *The radical r_2 coincides with the Kertész radical.*

Remark. The Kertész radical, which we shall denote by k, is discussed in [2; 3]. For an arbitrary $M\in{}_A\mathcal{M}$ it is defined by $k(M)=\{x\in M\,|\,Ax\subset\Phi(M)\}$. Using the easily verifiable fact that $L\in\Sigma^2_M$ if and only if M/L is irreducible, we note that our theorem is already partially covered in [2; 3], where modules which are not k-radical are being characterized. We give here a complete proof within the framework of our approach.

Proof. If M has no maximal submodules, then also $\Sigma^2_M=\emptyset$, so that $r_2(M)=$ $=k(M)=M$. In the other alternative we consider the set $\{L_\gamma|\gamma\in C\}$ of all maximal submodules of M and the corresponding set $\{\bar{L}_\gamma|\gamma\in C\}$. (See 3.3.) For each $\gamma\in C$ we have that $\bar{L}_\gamma$ is a submodule of M, that $L_\gamma\subset\bar{L}_\gamma\subset M$ and hence (by the maximality of L_γ) that $L_\gamma=\bar{L}_\gamma$ or $\bar{L}_\gamma=M$. Since $x\in k(M)$ if and only if $Ax\subset L_\gamma$ for all γ, and since the latter condition is equivalent with $x\in\bar{L}_\gamma$ for all γ, we have that

$$k(M)=\bigcap_\gamma \bar{L}_\gamma. \tag{2}$$

Furthermore, 3.3 (iii) shows that $L_\gamma\in\Sigma^2_M$ if and only if $L_\gamma=\bar{L}_\gamma$. Hence if $\bar{L}_\gamma=M$ for all γ, then $\Sigma^2_M=\emptyset$ and, applying (2), we obtain $r_2(M)=M=\bigcap_\gamma\bar{L}_\gamma=k(M)$. On the other hand, if at least one $\bar{L}_\gamma\neq M$, i.e. $\bar{L}_\gamma=L_\gamma$, then the redundancy of the (possible) M's in $\bigcap_\gamma\bar{L}_\gamma$ in (2) shows that $k(M)=\bigcap_{L\in\Sigma^2_M}L=r_2(M)$. This completes the proof.

Our final application of theorem 2.4 is the construction of one more radical based on a concept of 'modularity' of maximal submodules in the following sense.

3.7 Definition. A maximal submodule L of a module $M\in{}_A\mathcal{M}$ is called a Σ^3-*submodule* of M if there exists an $a\in A$ such that $ax-x\in L$ for all $x\in M$.

It easily follows that $\Sigma^3_M\subset\Sigma^2_M$, and if A has a unity element e, the reverse inclusion also holds; for in this case, if $L\in\Sigma^2_M$ we have that $ex-x\in r_1(M)\subset L$ for all $x\in M$. Thus we have:

3.8 Lemma. *For each* $M \in {}_A\mathcal{M}$: (i) $\Sigma_M^3 \subset \Sigma_M^2 \subset \Sigma_M^1$; (ii) *if A has a unity element then* $\Sigma_M^2 = \Sigma_M^3$.

The validity of the following auxilliary result may also be checked directly.

3.9 Lemma. Σ^3 *is a stable transferring isolator on* ${}_A\mathcal{M}$.

Denoting the radical associated with Σ^3 by r_3 we may state the following immediate consequence of lemma 3.8.

3.10 Corollary. *For each* $M \in {}_A\mathcal{M}$: (i) $r_1(M) \subset r_2(M) \subset r_3(M)$; (ii) *if A has a unity element then* $r_2(M) = r_3(M)$.

Regarding the second part of this corollary we note that in the case of a unitary left A-module M the radicals r_2 and r_3 coincide with the radical usually employed in this case, namely $\Phi(M)$. This also shows that r_3 is not idempotent.

For an arbitrary $M \in {}_A\mathcal{M}$, (A arbitrary), the stableness of Σ^3 has the implication that $L \in \Sigma_M^3$ if and only if $0 \in \Sigma_{M/L}^3$. This means that for $L \in \Sigma_M^3$ the factor module M/L has a 'quasi-unitary' property in the sense that there is an $a \in A$ such that $a(x+L) = x+L$ for all $x+L \in M/L$. Since L is maximal we have the additional property that M/L is irreducible. Calling a module $N \in {}_A\mathcal{M}$ *quasi-unitary* if there exists an element $a \in A$ with $ax = x$ for all $x \in N$, we have the following result in view of 2.5.

3.11 Corollary. *A non-zero module M in* ${}_A\mathcal{M}$ *is isomorphic to a subdirect product of quasi-unitary irreducible modules in* ${}_A\mathcal{M}$ *if and only if* $r_3(M) = 0$.

We observe that if A has a unity element, the r_i-semisimple modules, $(i=1, 2, 3)$, are in fact unitary, for $r_i(M) = 0$ implies that $M = M/r_i(M) = M/r_1(M) \cong AM$. Finally in this section we must mention that the exact relationship between r_2 and r_3 has not yet been settled. The probability seems to weigh in the direction of a difference.

§ 4. The module radicals r_i of ${}_A\mathcal{M}$ confined to the ground ring A

We conclude our discussion by comparing the module radicals $r_i(A)$, $(i=1, 2, 3)$, with ring radicals of A. The Baer lower radical β, the Jacobson radical J and the Brown—McCoy radical B were the best known candidates for this purpose.

4.1 Theorem. *For every associative ring* A: (i) $r_1(A) \subset \beta(A)$, (ii) $(r_2 A) \subset J(A)$, (iii) $r_3(A) \subset B(A)$.

Proof. We need only prove (i) and (iii) since (ii) has already been established in [3, 7]. (i) Let P be a prime ideal of A and let $x \in A$ with $Ax \subset P$. Each element of the ideal product $A(x)$ may be written as a finite sum of elements of the form $ax + \Sigma a_i x a_i'$, where $a, a_i, a_i' \in A$. Since $Ax \subset P$ we therefore have that $A(x) \subset P$ and since P is a prime ideal we obtain $x \in P$. This shows that $P \in \Sigma_A^1$, and we may conclude that $r_1(A) \subset \beta(A)$. (iii) Clearly, every modular maximal ideal of A, (cf. [5]), belongs to Σ_A^3. Hence $r_3(A) \subset B(A)$. This completes the proof.

In the case where A has a unity element it was already noted that $r_2(M)=r_3(M)$ for all $M\in {}_A\mathcal{M}$. Considering the module $A\in {}_A\mathcal{M}$ in this case, we observe that Σ_A^2 and Σ_A^3 coincide with the set of modular maximal left ideals of A, so that $r_2(A)= =r_3(A)=J(A)$. If A is a commutative ring (with or without unity element), we know that $J(A)=B(A)$. Moreover, the modular maximal ideals of A are exactly the Σ^3-ideals of A. Finally in this case, if $L\in\Sigma_A^2$, then $A^2\not\subset L$ and hence a result in [4] ensures the modularity of L. Thus if A is a commutative ring we have that $r_2(A)= =r_3(A)=J(A)=B(A)$.

Concerning r_1 no positive results are ensured by either of the above mentioned 'natural' conditions on A. If A has a unity element then $r_1(A)=0$, and in the case of commutative rings the Zassenhaus example A mentioned in [1], p. 20, supplies a counter example. Here $\beta(A)=A$, while $A^2=A$ shows that $r_1(A)=A_1\neq A$.

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