Solution of a problem of B. de la Rosa

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1. Introduction

In a previous paper [1] B. DE LA ROSA has introduced the radicals r_1, r_2 and r_3 for general left A-modules, where A is an associative ring, without any assumptions on commutativity or the existence of a unity element. For the definitions of these radicals and other concepts we refer to his paper. B. de la Rosa conjectures that, in general, $r_2 \neq r_3$. The purpose of this paper is to show that this is indeed the case.

2. Construction of the ring A

In order to prove that $r_2 \neq r_3$, we use a ring A, such that $r_2(A) \neq A$ but $r_3(A) = A$. The ring A is constructed similarly as the Ω-ring A used by F. A. SZÁSZ [2]. Let K be a field and Γ be an index set, where $|\Gamma| \geqq \aleph_0$. A is an algebra over K with the basic elements $a_\alpha, r_{\beta\gamma}, s_{\varepsilon\eta\vartheta} (\alpha, \beta, \gamma, \varepsilon, \eta, \vartheta \in \Gamma)$. Any element a in A has the form

$$(*) \qquad a = \sum_i^* \pi_i a_{\alpha_i} + \sum_{i,j}^* e_{ij} r_{\alpha_i \beta_j} + \sum_{i,j,k}^* \sigma_{ijk} s_{\alpha_i \beta_j \gamma_k},$$

where $\pi_i, \varrho_{ij}, \sigma_{ijk} \in K$ and all three sums Σ^* are finite.

The multiplication of the basic elements is defined according to the tabel

	a_ε	$r_{\varepsilon\eta}$	$s_{\varepsilon\eta\vartheta}$
a_α	a_α	$s_{\alpha\varepsilon\eta}$	$s_{\alpha\eta\vartheta}$
$r_{\alpha\beta}$	$\delta_{\beta\varepsilon} a_\alpha$	$\delta_{\beta\varepsilon} r_{\alpha\eta}$	$\delta_{\beta\varepsilon} s_{\alpha\eta\vartheta}$
$s_{\alpha\beta\gamma}$	$\delta_{\gamma\varepsilon} a_\alpha$	$\delta_{\gamma\varepsilon} s_{\alpha\beta\eta}$	$\delta_{\gamma\varepsilon} s_{\alpha\eta\vartheta}$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Because of the multiplication table one can show that the multiplication in A is associative. Let L be the subalgebra of A, generated by all elements $r_{\beta\gamma}, s_{\varepsilon\eta\vartheta}$. Without the operator set K, we get that A is an associative ring and L is a left ideal in A.

The left A-modules are the left ideals in A, so L is a left A-module. First we show that L a Σ^1-submodule of A is, i.e. if $x \in A$ and $Ax \subset L$, then $x \in L$ (cf. [1], definition

3.1). Let $Ax \subset L$ for an element $x \in A$ and suppose that $x \notin L$. Then, in the representation (*) of x, at least one of the π_i is $\neq 0$ $(\pi_i \in K)$. It follows that $(\pi_i^{-1} r_{\beta\alpha_i})x = a_\beta + l'$ and, since $Ax \subset L$, $a_\beta + l' \in L$. Hence $a_\beta \in L$, which is impossible. So $x \in L$ and L is a Σ^1-submodule of A: $L \in \Sigma_A^1$. Next we prove that L is a Σ^1-maximal submodule of A, i.e. a Σ^2-submodule of A. In fact we show that L is a maximal left ideal in A. Let $a \notin L$, then we prove that $La + L = A$. This shows that the left ideal in A, generated by L and a, is A. Since $a \notin L$, there exists an element $\pi_i \neq 0$, $\pi_i \in K$, in the representation (*) of a. Take an arbitrary element $\beta \in \Gamma$ and an arbitrary element $e \in K$. Then $\pi_i^{-1} e^r \beta \alpha_i a = e a_\beta + l''$, $l'' \in L$, or $e a_\beta = \pi_i^{-1} e^r \beta \alpha_i a - l'' \in La + L$. So any element of A belongs to $La + L$ or $A = La + L$. Then L is a Σ^2-submodule of A or $L \in \Sigma_A^2$.

Since $r_2(A) = \bigcap_{S \in \Sigma_A^2} S$ if $\Sigma_A^2 \neq \emptyset$, we get that $r_2(A) \subseteq L$ and $r_2(A) \neq A$.

A maximal left ideal L of the ring A is a Σ^3-submodule of A if there exists an $a \in A$ such that $ax - x \in L$ for all $x \in A$. Now we show that L is not a Σ^3-submodule of A or that $L \notin \Sigma_A^3$.

Suppose then that there exists an element $a \in A$ such that $ax - x \in L$ for all $x \in A$. Then $y(ax - x) \in L$ for all $x, y \in A$ or $[y(a-1)]x \in L$ for all $x, y \in A$.

a) If $a \in L$ then $a = \sum_{i,j}^* e_{ij} r_{\alpha_i \beta_j} + \sum_{i,j,k}^* \sigma_{ijk} s_{\alpha_i \beta_j \gamma_k}$. Since $\sum_{i,j}^*$ and $\sum_{i,j,k}^*$ are finite sums and $|\Gamma|$ is infinite, there exists an index $\varepsilon \in \Gamma$ such that $\varepsilon \neq \beta_j$, $\varepsilon \neq \gamma_k$ for all β_j, γ_k in the representation of a. Hence

$$a a_\varepsilon = \sum_{i,j}^* e_{ij} \delta_{\beta_j \varepsilon} a_{\alpha_i} + \sum_{i,j,k}^* \sigma_{ijk} \delta_{\gamma_k \varepsilon} a_{\alpha_i} = 0,$$

so $a a_\varepsilon - a_\varepsilon = -a_\varepsilon \in L$. Contradiction.

b) If $a \notin L$ then there exists an element $\pi_i \in K$, $\pi_i \neq 0$ in the representation of a. Choose $y = \pi_i^{-1} r_{\beta\alpha_i}$ in A, then $y(a-1) = a_\beta + l$, $l \in L$. Therefore $(a_\beta + l)x \in L$ for all $x \in A$. As $l \in L$, one can find, exactly as in case (a), an index $\varepsilon \in \Gamma$ such that $l a_\varepsilon = 0$. Then $(a_\beta + l) a_\varepsilon = a_\beta \in L$, which is a contradiction. So $L \notin \Sigma_A^3$.

This last result enables us to show that $\Sigma_A^3 = \emptyset$. Suppose, on the contrary, that $L' \in \Sigma_A^3$ i.e. L' is a maximal left ideal in A and there exists an element $a \in A$ such that $ax - x \in L'$ for all $x \in A$. Let a have the representation (*). Since $|\Gamma|$ is infinite and all sums Σ^*, occuring in the representation (*), are finite, we can choose an index $\varepsilon \in \Gamma$ such that $\varepsilon \neq \alpha_i$, $\varepsilon \neq \beta_j$ and $\varepsilon \neq \gamma_k$ for any choice of α_i, β_j or γ_k in (*).

Take $x = r_{\varepsilon\eta}$ in A (η is arbitrary in Γ). Then $a r_{\varepsilon\eta} - r_{\varepsilon\eta} \in L'$. Also $\left(\sum_{i,j}^* e_{ij} r_{\alpha_i \beta_j}\right) r_{\varepsilon\eta} = \sum_{i,j}^* e_{ij} r_{\alpha_i \beta_j} r_{\varepsilon\eta} = \sum_{i,j}^* e_{ij} \delta_{\beta_j \varepsilon} r_{\alpha_i \eta} = 0$. Similarly $\left(\sum_{i,j,k}^* \sigma_{ijk} s_{\alpha_i \beta_j \gamma_k}\right) r_{\varepsilon\eta} = 0$. And $\left(\sum_i^* \pi_i a_{\alpha_i}\right) r_{\varepsilon\eta} = \sum_i^* \pi_i a_{\alpha_i} r_{\varepsilon\eta} = \sum_i^* \pi_i s_{\alpha_i \varepsilon\eta}$. Hence $a\ r_{\varepsilon\eta} - r_{\varepsilon\eta} \in L'$ implies that $-r_{\varepsilon\eta} + \sum_i^* \pi_i s_{\alpha_i \varepsilon\eta} \in L'$ for all $\eta \in \Gamma$. Then take $r_{\lambda\varepsilon}$, where λ is arbitrarily chosen (in Γ). It follows that

$$(r_{\lambda\varepsilon})\left(-r_{\varepsilon\eta} + \sum_i^* \pi_i s_{\alpha_i \varepsilon\eta}\right) = -r_{\lambda\eta} + \sum_i^* \pi_i r_{\lambda\varepsilon} s_{\alpha_i \varepsilon\eta} = -r_{\lambda\eta} \in L' \quad \text{for all} \quad \lambda, \eta \in \Gamma.$$

Hence $r_{\lambda\eta} \in L'$ for all $\lambda, \eta \in \Gamma$.

Now choose $x = s_{\varepsilon\mu\vartheta}$, then $a s_{\varepsilon\mu\vartheta} - s_{\varepsilon\mu\vartheta} \in L'$. Using the representation (*) of a and proceeding as above, one gets that $-s_{\varepsilon\mu\vartheta} + \sum_i^* \pi_i s_{\alpha_i\mu\vartheta} \in L'$ for all $\mu, \vartheta \in \Gamma$.

Hence $(r_{\nu\varepsilon})(-s_{\varepsilon\mu\vartheta}+\sum_i^* \pi_i a_{\alpha_i\mu\vartheta})=-s_{\nu\mu\vartheta}\in L'$ for all $\nu, \mu, \vartheta\in\Gamma$. So $s_{\nu\mu\vartheta}\in L'$ for all $\nu, \mu, \vartheta\in\Gamma$, Then the subalgebra of A, generated by all $r_{\beta\gamma}, s_{\varepsilon\eta\vartheta}$, belongs to L', i.e. $L\subseteqq L'$. However L, L' are maximal left ideals, so $L=L'$. Then $L\in\Sigma_A^3$, which is a contradiction. This shows that $L'\in\Sigma_A^3$ does not exists, so $\Sigma_A^3=\emptyset$. By definition, $r_3(A)=A$ ([1], definition 2.2).

We have seen that $r_2(A)\neq A$, so, in general, $r_2\neq r_3$.

Literature

[1] B. DE LA ROSA, Concrete radicals in general modules.

[2] F. A. SZÁSZ, Lösung eines Problems bezüglich einer Charakterisierung des Jacobsonschen Radikals, *Acta Math. Acad. Sci. Hung.* **18** (3—4), (1967), 261—272.

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