The separator of a subset of a semigroup

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1. Introduction

In this paper we introduce a new notion by the help of the idealizator. This new notion is the separator of a subset of a semigroup. We investigate the properties of the separator in an arbitrary semigroup and characterize the unitary subsemigroups and the prime ideals by the help of their separators. We give conditions which imply that a maximal ideal is prime. The last section of this paper treats the separator of a free subsemigroup of a free semigroup.

Notations: If A is a subset of a semigroup S, then $\bar{A}$ denotes the subset $S \setminus A$. For other symbols we refer to [2].

2. The definition and basic properties of the separator

Let S be a semigroup and A any subset of S. As known [1], the idealizator of A is the set of all the elements x of S which satisfy the following conditions: $Ax \subseteq A$, $xA \subseteq A$. The idealizator of a subset A is denoted by $\mathrm{Id}\, A$.

If A is an empty set, then $\mathrm{Id}\, A$ equals to S. It is evident, that $\mathrm{Id}\, A$ is either empty or a subsemigroup of S.

Definition 1. Let S be a semigroup and A any subset of S. Then $\mathrm{Id}\, A \cap \mathrm{Id}\, \bar{A}$ will be called the *separator* of A and denoted by $\mathrm{Sep}\, A$.

In other words: An element x of S belongs to $\mathrm{Sep}\, A$ $(A \subseteq S)$ if and only if $xA \subseteq A$, $Ax \subseteq A$, $x\bar{A} \subseteq \bar{A}$, $\bar{A}x \subseteq \bar{A}$.

Remark 1. For any subset A of S, $\mathrm{Sep}\, A$ is either empty or a subsemigroup of S. Moreover, $\mathrm{Sep}\, A = \mathrm{Sep}\, \bar{A}$. In particular, $\mathrm{Sep}\, \square = \mathrm{Sep}\, S = S$.

Remark 2. If S is a semigroup with identity, then the identity element belongs to $\mathrm{Sep}\, A$ for any subset A of S.

Remark 3. If R is an ideal of a semigroup S, then $R \cap \mathrm{Sep}\, R = \square$.

Theorem 1. *Let $\{A_f : f \in F\}$ be any family of subsets of a semigroup S. Then $\cap \{\mathrm{Sep}\, A_f : f \in F\} \subseteq \mathrm{Sep} \cup \{A_f : f \in F\}$ and $\cap \mathrm{Sep}\, \{A_f : f \in F\} \subseteq \mathrm{Sep} \cap \{A_f : f \in F\}$.*

PROOF. Let $t \in \bigcap \{\text{Sep } A_f \colon f \in F\}$. Then $t \cdot [\bigcup \{A_f \colon f \in F\}] = \bigcup \{tA_f \colon f \in F\} \subseteq \bigcup \{A_f \colon f \in F\}$ and $t[\overline{\bigcup \{A_f \colon f \in F\}}] = t[\bigcap \{\overline{A}_f \colon f \in F\}] = \bigcap \{t\overline{A}_f \colon f \in F\} \subseteq \bigcap \{\overline{A}_f \colon f \in F\} = \overline{\bigcup \{A_f \colon f \in F\}}$.

Similarly, $[\bigcup \{A_f \colon f \in F\}] t \subseteq \bigcup \{A_f \colon f \in F\}$ and $[\overline{\bigcup \{A_f \colon f \in F\}}] \cdot t \subseteq \overline{\bigcup \{A_f \colon f \in F\}}$. Consequently, $t \in \text{Sep} \bigcup \{A_f \colon f \in F\}$ and the first part of the theorem is proved.

Since $\bigcap \{\text{Sep } A_f \colon f \in F\} = \bigcap \{\text{Sep } \overline{A}_f \colon f \in F\} \subseteq \text{Sep} \bigcup \{\overline{A}_f \colon f \in F\} = \text{Sep } \overline{\bigcap \{A_f \colon f \in F\}} = \text{Sep} \bigcap \{A_f \colon f \in F\}$, the theorem is proved.

Corollary 1. $\text{Sep } A \cap \text{Sep Sep } A \subseteq \text{Sep } (A \cup \text{Sep } A)$ for any subset A of a semigroup.

Theorem 2. *If A is a subsemigroup of a semigroup, then $A \cup \text{Sep } A$ is so.*

PROOF. Let A be a subsemigroup of a semigroup S. We may assume that $\text{Sep } A \neq \square$. Let $x, y \in A \cup \text{Sep } A$. Since $\text{Sep } A$ is also a subsemigroup of S, we have to consider only the case when $x \in A$ and $y \in \text{Sep } A$. Then xy and yx belong to A by the definition of the separator.

Theorem 3. *If A is a subset of a semigroup S such that $\text{Sep } A \neq \square$, then either $\text{Sep } A \subseteq A$ or $\text{Sep } A \subseteq \overline{A}$.*

PROOF. Let A be a subset of a semigroup S and assume that $A \cap \text{Sep } A \neq \square$. Let x be an arbitrary element of $\text{Sep } A$ and a an element of $A \cap \text{Sep } A$. Then $xa \in A$. If x were in $\overline{A}$, then xa would be in $\overline{A}$, too (because $a \in \text{Sep } A$), and this would be a contradiction. Consequently, if $A \cap \text{Sep } A \neq \square$ and $x \in \text{Sep } A$, then $x \in A$. In other words, $A \cap \text{Sep } A \neq \square$ implies that $\text{Sep } A \subseteq A$. Thus the theorem is proved.

Theorem 4. *Let Φ be a homomorphism of a semigroup S onto itself and R_1, R_2 subsemigroups of S such that $\Phi^{-1}(R_2) = R_1$. Then $\Phi(\text{Sep } R_1) = \text{Sep } R_2$.*

PROOF. Let x be an arbitrary element of $\text{Sep } R_1$. We prove that $\Phi(x)$ belongs to $\text{Sep } R_2$. For this porpose, let y be an arbitrary element of R_2. Then there exists a $y_0 \in S$ such that $\Phi(y_0) = y$. Assume $\Phi(x) y \in \overline{R}_2$. Then $\Phi(x)\Phi(y_0) = \Phi(xy_0) \in \overline{R}_2$ and therefore $xy_0 \in \overline{R}_1$. Thus $y_0 \in \overline{R}_1$, since $x \in \text{Sep } R_1$. Hence $y = \Phi(y_0) \in \overline{R}_2$ which contradicts the condition that $y \in R_2$. Consequently, $\Phi(x) y$ and, similarly, $y\Phi(x) \in R_2$ for every $x \in \text{Sep } R_1, y \in R_2$.

Now let z be an arbitary element of $\overline{R}_2$. There exists a $z_0 \in S$ such that $\Phi(z_0) = z$. Assume $\Phi(x) z \in R_2$. Then $\Phi(x)\Phi(z_0) = \Phi(xz_0)$ belongs to R_2 and therefore $xz_0 \in R_1$. Thus $z_0 \in R_1$, since $x \in \text{Sep } R_1$. Hence $z = \Phi(z_0) \in R_2$ which contradicts the assumption, that $z \in \overline{R}_2$. Consequently $\Phi(x) z \in \overline{R}_2$ and, similarly, $z\Phi(x) \in \overline{R}_2$. Thus $\Phi(x)$ belongs to $\text{Sep } R_2$ indeed, that is $\Phi(\text{Sep } R_1) \subseteq \text{Sep } R_2$.

We complete the proof by showing that $t \notin \text{Sep } R_1$ implies $\Phi(t) \notin \text{Sep } R_2$. In fact, if $t \notin \text{Sep } R_1$, then either there exists an element u in R_1 such that at least one of the relations $tu \in \overline{R}_1$ and $ut \in \overline{R}_1$ holds or there exists an element v in $\overline{R}_1$ such that at least one of $tv \in R_1$ and $vt \in R_1$ holds. Consider the case when $u \in R_1$ and $tu \in \overline{R}_1$ (the other cases can be discussed similarly). Then $\Phi(u) \in R_2$ and $\Phi(t)\Phi(u) \in \overline{R}_2$ which mean that $\Phi(t) \notin \text{Sep } R_2$, indeed.

Corollary 2. *If Φ is an isomorphism of a semigroup S onto itself and A and B are subsemigroup of S such that $\Phi(A) = B$, then $\Phi(\text{Sep } A) = \text{Sep } B$.*

3. Separator including and separator excluding subsets

Definition 2. A subset A of a semigroup is said to be *separator including (excluding)* if $\mathrm{Sep}\, A \subseteq A$ ($\mathrm{Sep}\, A \subseteq \bar{A}$). In the case of $\mathrm{Sep}\, A = \square$, the subset A will be considered separator including as well as separator excluding.

Remark 4. If a subset A of a semigroup is separator including (excluding), then, evidently, $\bar{A}$ is separator excluding (including). In particular, the semigroup S is a separator including subset of itself and the empty set is a separator excluding one.

Remark 5. If A and B are separator including subsemigroups of a semigroup S and Φ is a homomorphism of A onto B, then, in general, $\Phi(\mathrm{Sep}\, A) \neq \mathrm{Sep}\, B$. But if Φ is a homomorphism of A onto B such that $\mathrm{Sep}\, A = \Phi^{-1}(\mathrm{Sep}\, B)$, then Φ is a homomorphism of $\mathrm{Sep}\,\mathrm{Sep}\, A$ onto $\mathrm{Sep}\,\mathrm{Sep}\, B$.

Example: Consider the semigroup $S(\cdot)$ of four elements in which the operation $(\cdot)$ is given by the following Cayley table:

	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

In this semigroup $\mathrm{Sep}\,\{1\} = \{1\}$, $\mathrm{Sep}\,\{a, b, 1\} = \{a, b, 1\}$. $\mathrm{Sep}\,\{a\} = \{1\}$, thus $\mathrm{Sep}\,\{a\} \cap \{a\} = \square$ and $\mathrm{Sep}\,\{a\} \cup \{a\} \neq S$. $\mathrm{Sep}\,\{0, a. b\} = \{1\}$, thus $\mathrm{Sep}\,\{0, a, b\} \cap \{0, a, b\} = \square$ and $\mathrm{Sep}\,\{0, a, b\} \cup \{0, a, b\} = S$.

Theorem 5. *Let A be a separator including (excluding) subset and B an arbitrary subset of a semigroup S such that $\mathrm{Sep}\, A \cap \mathrm{Sep}\, B \neq \square$. Then $A \cup B$ ($A \cap B$) also is a separator including (excluding) subset of S and its separator is non-empty.*

Proof. By Theorem 3, our assertion concerning a separator including A will be proved if we show that the intersection of $A \cup B$ and $\mathrm{Sep}\,(A \cup B)$ is non-empty. Using Theorem 1 and the condition that $\mathrm{Sep}\, A \subseteq A$ we get

$$(A \cup B) \cap \mathrm{Sep}\,(A \cup B) \supseteq A \cap \mathrm{Sep}\, A \cap \mathrm{Sep}\, B = \mathrm{Sep}\, A \cap \mathrm{Sep}\, B$$

and the last term is non-empty, by one of the conditions. Consequently, $(A \cup B) \cap \mathrm{Sep}\,(A \cup B) \neq \square$, indeed. Consider the assertion concerning a separator excluding A. Then $\bar{A}$ is separator including. Since $\mathrm{Sep}\, \bar{A} \cap \mathrm{Sep}\, \bar{B} = \mathrm{Sep}\, A \cap \mathrm{Sep}\, B \neq \square$ by our assertion proved just now, $\bar{A} \cup \bar{B}$ is separator including and $\mathrm{Sep}\,(\bar{A} \cup \bar{B}) \neq \square$.

Consequently, $A \cap B = \overline{\bar{A} \cup \bar{B}}$ is separator excluding and $\square \neq \mathrm{Sep}\,(\bar{A} \cup \bar{B}) = \mathrm{Sep}\,(\overline{\bar{A} \cup \bar{B}}) = \mathrm{Sep}\,(A \cap B)$.

Corollary 3. *Let S be a semigroup and $A, B \subseteq S$. If A is a separator including subset and B is a separator excluding one such that $\mathrm{Sep}\, A \cap \mathrm{Sep}\, B \neq \square$, then $A \cup B$ is separator including, $A \cap B$ is separator excluding and $\mathrm{Sep}\,(A \cup B) \neq \square$, $\mathrm{Sep}\,(A \cap B) \neq \square$.*

Theorem 6. *Let A and B separator including (excluding) subsets of a semigroup S such that* $\text{Sep}\, A \cap \text{Sep}\, B \neq \square$. *Then $A \cap B$ $(A \cup B)$ also is a separator including (excluding) subset of S and its separator is non-empty.*

Proof. By Theorem 3, our assertion concerning a separator including A and B will be proved if we show that the intersection of $A \cap B$ and $\text{Sep}\,(A \cap B)$ is non-empty. Using Theorem 1 and the condition that $\text{Sep}\, A \subseteq A$ and $\text{Sep}\, B \subseteq B$ we get

$$A \cap B \cap \text{Sep}\,(A \cap B) = A \cap B \cap \text{Sep}\,(\overline{A \cap B}) = A \cap B \cap \text{Sep}\,(\bar{A} \cup \bar{B}) \supseteq$$

$$A \cap B \cap \text{Sep}\,\bar{A} \cap \text{Sep}\,\bar{B} = \text{Sep}\, A \cap \text{Sep}\, B$$

and the last term is non-empty, by one of the conditions. Consequently, $A \cap B \cap \text{Sep}\,(A \cap B) \neq \square$, indeed. Consider the assertion concerning separator excluding A and B. Then $\bar{A}$ and $\bar{B}$ are separator including. Since $\text{Sep}\,\bar{A} \cap \text{Sep}\,\bar{B} = \text{Sep}\, A \cap \text{Sep}\, B \neq \square$, $\bar{A} \cap \bar{B}$ is separator including and $\text{Sep}\,(\bar{A} \cap \bar{B}) \neq \square$. Consequently, $A \cup B = \overline{\bar{A} \cap \bar{B}}$ is separator excluding and $\square \neq \text{Sep}\,(\bar{A} \cap \bar{B}) = \text{Sep}\,(\overline{\bar{A} \cap \bar{B}}) = \text{Sep}\,(A \cup B)$.

Corollary 4. *Let A be a separator including subset and B an arbitrary subset of a semigroup S. If* $\text{Sep}\, A \cap \text{Sep}\, B \neq \square$, *then $B \setminus A$ is separator excluding and* $\text{Sep}\,(B \setminus A) \neq \square$, *where $B \setminus A$ denotes the subset $B \cap \bar{A}$. Obviously, $B = (B \setminus A) \cup (B \cap A)$ holds for every subsets A and B of S. Hence, if A is separator including, B separator excluding and* $\text{Sep}\, A \cap \text{Sep}\, B \neq \square$, *then B is the union of two separator excluding subsets (because $B \setminus A$ and $B \cap A$ are separator excluding).*

Theorem 7. *The separator of any subset A of a semigroup is separator including (that is,* $\text{Sep}\,\text{Sep}\, A \subseteq \text{Sep}\, A$*), provided that* $\text{Sep}\, A \neq \square$.

Proof. Let A be a subset of a semigroup. We may assume that $\text{Sep}\,\text{Sep}\, A \neq \square$. Let t be an arbitrary element of $\text{Sep}\,\text{Sep}\, A$. We prove $t \in \text{Sep}\, A$. Let a be an arbitrary element of A. Then, for every $x \in \text{Sep}\, A$, $xt \in \text{Sep}\, A$ and thus $xta \in A$. If ta were in $\bar{A}$, then xta would be in $\bar{A}$, because $x \in \text{Sep}\, A$, and this would be a contradiction. Hence, $ta \in A$. Similarly, $at \in A$. Now, let b be an arbitrary element of $\bar{A}$. We prove that $tb \in \bar{A}$. Let x be an arbitrary element of $\text{Sep}\, A$. Then $xt \in \text{Sep}\, A$ and thus $xtb \in \bar{A}$. If tb were in A, then xtb would be in A because $x \in \text{Sep}\, A$ and this would be a contradiction. Hence, $tb \in \bar{A}$, indeed. Similarly, $bt \in \bar{A}$. Thus, by the definition of the separator, t belongs to $\text{Sep}\, A$. This means that $\text{Sep}\,\text{Sep}\, A \subseteq \text{Sep}\, A$.

Corollary 5. *If the separator of a subset A of a semigroup S is a minimal subsemigroup, then* $\text{Sep}\,\text{Sep}\, A$ *is either empty or equal to* $\text{Sep}\, A$.

4. Unitary subsemigroups and prime ideals

Definition 3. A subsemigroup U of a semigroup S is called *unitary* if $ab \in U$ and $a \in U$ imply $b \in U$ and $ab \in U$, $b \in U$ imply $a \in U$ for any elements a, b of S. In other words: A subsemigroup U of S is unitary if $ab \in U$ implies either $a, b \in U$ or $a, b \in \bar{U}$.

Theorem 8. *For a subsemigroup A of a semigroup S the following two assertions are equivalent:*

(A): $A = \text{Sep}\, A$,

(B): *A is a unitary subsemigroup.*

PROOF. (A) implies (B): Let A be a subsemigroup such that $A=\text{Sep}\, A$. If $a\in A$ $(=\text{Sep}\, A)$ and $b\in\bar{A}$ or $a\in\bar{A}$ and $b\in A$, then $ab\in\bar{A}$. Consequently, $ab\in A$ implies either $a, b\in A$ or $a, b\in\bar{A}$, indeed.

(B) implies (A): Let A be a unitary subsemigroup of S and $a\in A$. Then a $A\subseteq A$ and $Aa\subseteq A$ because A is a subsemigroup of S, moreover $a\bar{A}\subseteq\bar{A}$ and $\bar{A}a\subseteq\bar{A}$ because A is unitary. Hence, $A\subseteq\text{Sep}\, A$. Consequently, $A=\text{Sep}\, A$ by Theorem 3.

The following two theorems deal with prime ideals. The first of them, which is partly known [2], characterizes the prime ideals and the second gives conditions for a maximal ideal to be prime.

Theorem 9. *A subsemigroup of a semigroup is a prime ideal if and only if its complement is a unitary subsemigroup.*

PROOF. For any ideal P of a semigroup, $\text{Sep}\, P\subseteq\bar{P}$ by Remark 3 and $\bar{P}P\subseteq P$, $P\bar{P}\subseteq P$. Assume that the ideal P is prime. Then $\bar{P}\bar{P}\subseteq\bar{P}$, too, which implies that $\bar{P}\subseteq\text{Sep}\, P$. Consequently, $\bar{P}=\text{Sep}\, P=\text{Sep}\,\bar{P}$. By Theorem 8, $\bar{P}$ is a unitary subsemigroup.

Conversely, assume that P is a subsemigroup of a semigroup S and $\bar{P}$ is a unitary subsemigroup. Then $\bar{P}=\text{Sep}\,\bar{P}=\text{Sep}\, P$. Let x be an arbitrary element of S. If $x\in\bar{P}$, then $px\in P$ and $xp\in P$ $(p\in P)$ because $\bar{P}=\text{Sep}\, P$. If $x\in P$, then $xP\subseteq P$ and $Px\subseteq P$ because P is a subsemigroup. Thus $xP\subseteq P$ and $Px\subseteq P$, that is, P is an ideal. Since $\bar{P}$ is a subsemigroup by the one of the conditions, the ideal P is prime [2].

Theorem 10. *Let I be a maximal ideal of a semigroup S. If*

(i) *I^1 is a maximal subsemigroup of S^1 and*

(ii) Sep *I has a non identity element,*

then I is a prime ideal.

PROOF. Assume that I satisfies the conditions (i)—(ii), and let $e\in\text{Sep}\, I$, $e\neq 1$. Then, by Remark 3, $e\notin I^1$ and thus $I^1\cup\text{Sep}\, I\supset I^1$. It follows by Theorem 2 and Condition (i) that $I^1\cup\text{Sep}\, I=S^1$. If $S=S^1$, then $1\in\text{Sep}\, I$ and so $S=I^1\cup\text{Sep}\, I=I\cup\text{Sep}\, I$. If $S\neq S^1$, then the condition (i) implies that I is a maximal subsemigroup of S and, consequently, $I\cup\text{Sep}\, I=S$ also in this case. On the other hand, $I\cap\text{Sep}\, I=\square$, by Remark 3. Thus $\text{Sep}\,\bar{I}=\bar{I}$, that is, $\bar{I}$ is a unitary subsemigroup of S (see Theorem 8) and Theorem 9 implies that I is a prime ideal, indeed.

5. Separators in a free semigroup

In this section we shall make use several times the following lemma (see [2], § 9.1):

Lemma. *A subsemigroup T of a free semigroup S is a free subsemigroup if and only if $sT\cap T\neq\square$ and $Ts\cap T\neq\square$ together imply $s\in T$ for each element s of S.*

Theorem 11. *Any free subsemigroup of a free semigroup is separator including.*

PROOF. Let T be a free subsemigroup of a free semigroup S. We may assume that $\text{Sep}\, T\neq\square$. Let s be an arbitrary element of Sep T. Then $sT\subseteq T$ and $Ts\subseteq T$, that is $sT\cap T\neq\square$ and $Ts\cap T\neq\square$. Thus $s\in T$ by the Lemma. Consequently, $\text{Sep}\, T\subseteq T$.

Theorem 12. *The separator of any free subsemigroup T of a free semigroups is free, unless* Sep $T \neq \square$.

Proof. Assume that T is a free subsemigroup of a free semigroup S and Sep $T \neq \square$. Let s be an arbitrary element of S such that $s(\text{Sep } T) \cap \text{Sep } T \neq \square$ and $(\text{Sep } T)s \cap \text{Sep } T \neq \square$. Then $sT \cap T \neq \square$ and $Ts \cap T \neq \square$ since Sep $T \subseteq T$. Hence $s \in T$ since T is a free subsemigroup. We prove that $s \in \text{Sep } T$. First, $sT \subseteq T$ and $Ts \subseteq T$ because $s \in T$. Next, let t be an arbitrary element of $\overline{T}$. We have to show that st and ts belong to $\overline{T}$. By the condition $(\text{Sep } T)s \cap \text{Sep } T \neq \square$ there exist $m_1, m_2 \in \text{Sep } T (\subseteq T)$ such that $m_1 s = m_2$. Hence $m_2 t \in \overline{T}$. Now, if we suppose $st \in T$, then we get $m_2 t = (m_1 s)t = m_1(st) \in T$ which contradicts our preceding result. Consequently $s\overline{T} \subseteq \overline{T}$ and, similarly, $\overline{T}s \subseteq \overline{T}$. Thus $s \in \text{Sep } T$, as asserted, and we have show that the conditions $s(\text{Sep } T) \cap \text{Sep } T \neq \square$ and $(\text{Sep } T)s \cap \text{Sep } T \neq \square$ imply $s \in \text{Sep } T$ for each $s \in S$. It follows (see again the Lemma) that Sep T is a free subsemigroup of S.

Theorem 13. *For any subset $T \neq \square$ of a free semigroup S the following two assertions are equivalent.*

(A) *T is a free subsemigroup with the property that, for any elements $t \in T$ and $s \in S$, $tst \in T$ implies both $ts \in T$ and $st \in T$.*

(B) $T = \text{Sep } T$.

Proof. Assume that (A) holds. Then Sep $T \subseteq T$ by Theorem 11. We have to prove $T \subseteq \text{Sep } T$. Let t be an arbitrary element of T. Then $tT \in T$ and $Tt \subseteq T$. Let s be an arbitrary element of $\overline{T}$. We show that $ts \in \overline{T}$ and $st \in \overline{T}$. If we assume that $ts \in T$ or $st \in T$, then $tst \in T$ and thus, by (A), both $ts \in T$ and $st \in T$, that is $sT \cap T \neq \square$ and $Ts \cap T \neq \square$. Hence $s \in T$, since T is a free subsemigroup. This contradicts the condition that $s \in \overline{T}$. Consequently, $t\overline{T} \subseteq \overline{T}$ and $\overline{T}t \subseteq \overline{T}$. Thus $t \in \text{Sep } T$ and $T \subseteq \text{Sep } T$, indeed.

Now, let T be a non-empty subset of a free semigroup S and $T = \text{Sep } T$. Then T is a subsemigroup of S (see Remark 1). Let s be an arbitrary element of S such that $sT \cap T \neq \square$ and $Ts \cap T \neq \square$. Then $s \in T$, because $s \in \overline{T}$ would imply $sT = s(\text{Sep } T) = s(\text{Sep } \overline{T}) \subseteq \overline{T}$, that is $sT \cap T = \square$, in contradiction to our assumption $sT \cap T \neq \square$. Hence T is a free subsemigroup by the Lemma. Let t be an arbitrary element of T and s one of S such that $tst \in T$. Since $T = \text{Sep } T$, T is a unitary subsemigroup in S (see Theorem 8.) and thus $t \in T$ implies that both st and ts belong to T.

Theorem 14. *Let S be a free semigroup and A a subsemigroup of S such that* Sep $A \supseteq A^n$ *for some integer n. Then A is a free subsemigroup.*

Proof. Let s be any element of S such that $sA \cap A \neq \square$ and $As \cap A \neq \square$. Then there exists an element a of A such that $sa \in A$. Since A is a subsemigroup, $sa^n \in A$, too. Hence $s \in A$, because $a^n \in A^n \subseteq \text{Sep } A$. Consequently, A is a free subsemigroup of S.

References

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