An extension of the Hilbert basis theorem

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An analogue of the Hilbert basis theorem will be developed for semirings using the class of monic ideals studied by DALE and ALLEN [3]. Unlike the methods used by ALLEN [1], BRACKIN [2] and STONE [4], the method used here will not require the use of the Hilbert basis theorem for rings. The converse of this result as well as an application will also be given.

We will use the standard definitions and results appearing in [1] and [3]. The following are included for the convience of the reader: When R is a semiring with an identity, an ideal M in $R[x]$ will be called *monic* provided $f(x)=\Sigma a_i x^i \in M$ implies that $a_i x^i \in M$ for each i. When A is an ideal in $R[x]$ the set $\{a \in R|$ there exist $f(x) \in A$ such that a is the coefficient of the i^{th} term of $f(x)\}$ is denoted by A_i and the ascending chain of ideals $\{A_n\}$ will be called *coefficient ideals*. Z^+ will denote the semiring of nonnegative integers.

Lemma 1. *Let R be a semiring with an identity and A and B be the ideals in $R[x]$. If $A \subset B$, then $A_i \subset B_i$ for each $i \in Z^+$.*

PROOF. If $a \in A_i$, then a is the coefficient of the ith term of some $f(x) \in A \subset B$. Consequently, $a \in B_i$ and the result follows.

Corollary 2. *If R is a semiring with an identity where A and B are ideals in $R[x]$ with $A=B$, then $A_i=B_i$ for each $i \in Z^+$*

PROOF. If $A=B$, then $A \subset B$ and $B \subset A$ and it follows from Lemma 1 that $A_i \subset B_i$ and $B_i \subset A_i$ for each i. Consequently, $A_i=B_i$ for each $i \in Z^+$.

The following theorem is essential for the proof of the basis theorem for monic ideals. This theorem reveals an important fact about the coefficient ideals of an ascending chain of ideals in $R[x]$ when R is a Noetherian semiring.

Theorem 3. *If R is a Noetherian semiring and $\{A_t\}$ is an ascending chain of ideals in $R[x]$ where $\{A_{ti}\}$ are the coefficient ideals of A_t for each t, then there exist non-negative integers N and M such that $A_{Ni}=A_{ni}$ for each $n \geqq N$ and $i \in Z^+$, and $A_{tM}=A_{tm}$ for each $m \geqq M$ and $t \in Z^+$.*

PROOF. Since $\{A_{ti}\}$ are the coefficient ideals of A_t for each t and $\{A_t\}$ is an ascending chain of ideals in $R[x]$, a repeated application of Lemma 1 gives the following array of coefficient ideals in R;

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cup & & \cup & & \cup & & & & \cup \\
A_{\mu 0} & \subset & A_{\mu 1} & \subset & A_{\mu 2} & \subset & \dots & \subset & A_{\mu\nu} \subset \dots \\
\cup & & \cup & & \cup & & & & \cup \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\cup & & \cup & & \cup & & & & \cup \\
A_{20} & \subset & A_{21} & \subset & A_{22} & \subset & \dots & \subset & A_{2\nu} \subset \dots \\
\cup & & \cup & & \cup & & & & \cup \\
A_{10} & \subset & A_{11} & \subset & A_{12} & \subset & \dots & \subset & A_{1\nu} \subset \dots \\
\cup & & \cup & & \cup & & & & \cup \\
A_{00} & \subset & A_{01} & \subset & A_{02} & \subset & \dots & \subset & A_{0\nu} \subset \dots
\end{array}$$

It is clear that $\{A_{\varrho\varrho}\}$ is an ascending chain of ideals in R and it follows that there exists a non-negative integer τ such that $A_{\tau\tau}=A_{\varrho\varrho}$ if $\varrho\geqq\tau$, since R is a Noetherian semiring. Consider $A_{\varrho\nu}$ where $\varrho\geqq\tau$ and $\nu\geqq\tau$. If $\varrho=\nu$, then clearly $A_{\varrho\nu}=A_{\tau\tau}$. If $\varrho>\nu\geqq\tau$, then there exist non-negative integers α and μ such that $\nu=\tau+\alpha$ and $\varrho=\nu+\mu=\tau+(\alpha+\mu)$. Since ϱ is finite consider the following array.

$$\begin{array}{ccccccccc}
A_{\varrho\,\tau} & \subset & A_{\varrho\,\tau+1} & \subset \dots \subset & A_{\varrho\,\nu} & \subset \dots \subset & A_{\varrho\,\varrho} \\
\cup & & \cup & & \cup & & \cup \\
\vdots & & \vdots & & \vdots & & \vdots \\
\cup & & \cup & & \cup & & \cup \\
A_{\nu\,\tau} & \subset & A_{\nu\,\tau+1} & \subset \dots \subset & A_{\nu\,\nu} & \subset \dots \subset & A_{\nu\,\varrho} \\
\cup & & \cup & & \cup & & \cup \\
\vdots & & \vdots & & \vdots & & \vdots \\
\cup & & \cup & & \cup & & \cup \\
A_{\tau+1\,\tau} & \subset & A_{\tau+1\,\tau+1} & \subset \dots \subset & A_{\tau+1\,\nu} & \subset \dots \subset & A_{\tau+1\,\varrho} \\
\cup & & \cup & & \cup & & \cup \\
A_{\tau\,\tau} & \subset & A_{\tau\,\tau+1} & \subset \dots \subset & A_{\tau\,\nu} & \subset \dots \subset & A_{\tau\,\varrho}
\end{array}$$

From

$$A_{\tau\,\tau}\subset A_{\tau+1\,\tau}\subset A_{\tau+1\,\tau+1}\subset\dots\subset A_{\nu\,\nu-1}\subset A_{\nu\,\nu}\subset A_{\nu+1\,\nu}\subset\dots\subset A_{\varrho\,\varrho-1}\subset A_{\varrho\,\varrho}$$

and

$$A_{\tau\,\tau}=A_{\tau+1\,\tau+1}=A_{\tau+2\,\tau+2}=\dots=A_{\nu\,\nu}=\dots=A_{\varrho\,\varrho},$$

it follows that

(1) $$A_{\tau\,\tau}=A_{\tau+1\,\tau}=A_{\tau+2\,\tau+1}=\dots=A_{\varrho\,\varrho-1}$$

and consequently,

$$A_{\tau\,\tau}=A_{\nu+1\,\nu}.$$

Similarly from $A_{\tau+1\,\tau}\subset A_{\tau+2\,\tau}\subset A_{\tau+2\,\tau+1}\subset\dots\subset A_{\nu+1\,\nu-1}\subset A_{\nu+1\,\nu}\subset A_{\nu+2\,\nu}\subset\dots\subset A_{\varrho\,\varrho-2}\subset$ $\subset A_{\varrho\,\varrho-1}$ and equation (1) it follows that $A_{\tau\,\tau}=A_{\tau+2\,\tau}=A_{\tau+3\,\tau+1}=\dots=A_{\nu+2\,\nu}=\dots$ $\dots=A_{\varrho\,\varrho-2}$ and consequently, $A_{\tau\,\tau}=A_{\nu+2\,\nu}$. Continuing in this manner one obtains $A_{\tau\,\tau}=A_{\tau+\mu\,\tau}=A_{\tau+\mu+1\,\tau+1}=\dots=A_{\tau+\mu+\alpha\,\tau+\alpha}=A_{\varrho\,\nu}$ and consequently, $A_{\tau\,\tau}=A_{\varrho\,\nu}$. If $\nu>\varrho\geqq\tau$, then by an argument analogous to the above, it can be shown that $A_{\varrho\,\nu}=A_{\tau\,\tau}$. Consequently, $A_{\varrho\,\nu}=A_{\tau\,\tau}$ if $\varrho\geqq\tau$ and $\nu\geqq\tau$. Since only a finite number of columns appear before the τth column and R is Noetherian there exists n_t where

$t \in \{0, 1, 2, \dots, \tau-1\}$, such that $A_{mt} = A_{n_t t}$ for each $m \geqq n_t$. Letting $N = \max\{n_0, \dots, n_{\tau-1}, \tau\}$, it follows that $A_{Nt} = A_{nt}$ if $n \geqq N$ and $t \in Z^+$. Since only a finite number of rows appear before the τth row and R is Noetherian, it follows that there exists m_i where $i \in \{0, 1, 2, \dots, \tau-1\}$, such that $A_{im_i} = A_{in}$ for each $n \geqq m_i$. Letting $M = \max\{m_0, m_1, \dots, m_{\tau-1}, \tau\}$, it is clear that $A_{iM} = A_{im}$ for each $m \geqq M$ and $i \in Z^+$, and the result follows.

An interesting consequence of this theorem is that any ascending chain of ideals in $R[x]$, where R is a Noetherian semiring, can have only a finite number of coefficient ideals in R. It can also be shown by an argument analogous to the proof of the above theorem that if $\{A_{ni}\}$ is a double sequence of ideals in a Noetherian semiring R such that $\{A_{ni}\}$ is an ascending chain for each $n \in Z^+$ and $\{A_{ni}\}$ is an ascending chain for each $i \in Z^+$ then the sequence is necessarily finite. This result is very useful in the study of polynomial semirings over a Noetherian semiring.

Lemma 4. *Let R be a semiring with an identity where A and B are monic ideals in $R[x]$ such that $A \subset B$. If $A_i = B_i$ for each i, then $A = B$.*

Proof. If $f(x) = a_n x^n + \dots + a_0 \in B$, then $a_i \in B_i = A_i$ for each $i \in \{0, 1, 2, \dots, n\}$. Consequently, there exist polynomials $f_i(x) \in A$ such that a_i is the coefficient of the ith term of $f_i(x)$. Since A is monic, it follows that $a_i x^i \in A$, for each $i \in \{0, 1, 2, \dots, n\}$, and $f(x) \in A$. Therefore $B \subset A$ and it is clear that $A = B$.

It is now possible to prove the following analogue of the classical Hilbert Basis Theorem.

Theorem 5. *If R is a Noetherian semiring, then every ascending chain of monic ideals in $R[x]$ is finite.*

Proof. Let $\{A_n\}$ be an ascending chain of monic ideals in $R[x]$ and $\{A_{ni}\}$ the corresponding coefficient ideals in R for each A_n. Since R is a Noetherian semiring, it follows from Theorem 3 that there exists a positive integer N such that $A_{Nt} = A_{nt}$ for each $n \geqq N$ and $t \in Z^+$. Consequently, by Lemma 4 it is clear that $A_N = A_n$ for each $n \geqq N$ and the theorem follows.

The converse of this theorem will now be established; i.e., if $R[x]$ satisfies the ascending chain condition on monic ideals, then R is a Noetherian semiring. This will be obtained with the aid of the following lemma:

Lemma 6. *If R is a semiring with an identity and A and B are ideals in R, then $A[x] = B[x]$ if and only if $A = B$.*

Proof. It is clear that $A = B$ implies $A[x] = B[x]$. If $A[x] = B[x]$, then it follows from Corollary 2 that $A[x]_i = B[x]_i$ for each i. Since $A = A[x]_i$ and $B = B[x]_i$ foreach i, the result follows,

Theorem 7. *If R is a semiring with an identity such that every ascending chain of monic ideals in $R[x]$ is finite, then R is Noetherian.*

Proof. If $\{A_n\}$ is an ascending chain of ideals in R, it follows that $\{A_n[x]\}$ is an ascending chain of monic ideals in $R[x]$. Consequently, there exists a non-negative integer N such that $A_N[x] = A_n[x]$ if $n \geqq N$ and it follows from Lemma 6 that $A_N = A_n$ for each $n \geqq N$. Therefore, R is a Noetherian semiring.

Definition 8. An ascending sequence of ideals $\{A_n\}$ in $R[x]$, where R is a semiring with an identity, will be called a *para-monic* sequence of ideals provided: given any positive integer N there exists $m>N$ and a monic ideal B in $R[x]$ such that $A_m\subset B\subset A_{m+1}$.

The analogue of the Hilbert Basis Theorem with respect to monic ideals will not be used to prove the following theorem.

Theorem 9. *If R is a Noetherian semiring, then every para-monic sequence of ideals in $R[x]$ is finite.*

Proof. Let $\{A_n\}$ be a para-monic sequence of ideals in $R[x]$. If $N=1$ there is a positive integer $m_1>1$ and a monic ideal $B_1\subset R[x]$ such that $A_{m_1}\subset B\subset A_{m_1+1}$. If $N=m_1+1$, there is a positive integer $m_2>m_1+1$ and a monic ideal B_2 in $R[x]$ such that $A_{m_2}\subset B_2\subset A_{m_2+1}$ and it follows that $B_1\subset B_2$. If $N=m_2+1$, there is a positive integer $m_3>m_2+1$ and a monic ideal B_3 in $R[x]$ such that $A_{m_3}\subset B_3\subset A_{m_3+1}$ and it follows that $B_1\subset B_2\subset B_3$. Continuing in this manner, an ascending sequence of monic ideals $\{B_i\}$ is obtained. As a result of Theorem 5 there exists a positive integer M such that $B_M=B_r$ if $r\geqq M$. Therefore $A_M=A_r$ if $r\geqq M$ and the result follows.

References

[1] Paul J. Allen, An extension of the Hilbert basis theorem to semirings, *Publ. Math. (Debrecen)* 22 (1975), 31—34.

[2] Paul J. Allen and E. J. Brackin, A basis theorem for the semiring part of a Boolian Algebra, *Publ. Math. (Debrecen)* **20** (1973), 153—155.

[3] Louis Dale and Paul J. Allen, Ideal theory in polynomial semirings, *Publ. Math. (Debrecen)* **23** (1976), 183—190.

[4] H. E. Stone, Ideals in halfrings, *Proc. Amer. Math. Soc.* **33**, (1972), 8—14.

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