Characterization of the projectivity of principal ideals by generalized complements

By G. SZÁSZ (Budapest)

1. Introduction

In [4] the concept of complement was generalized in the following manner:

Definition 1. Let a, u, v be arbitrary elements of a lattice L. An element x of L is called a (u, v)-*complement* of a (or: a *generalized complement* of a with respect to u and v) if $a \cap x \leqq u$ and $a \cup x \geqq v$.

Definition 2. A lattice L is called *complemented in generalized sense* (or: *generalized complemented*) if, given arbitrary elements a, u, v of L, there exists at least one (u, v)-complement of a in L.

These generalized concepts were discussed also in the papers [2], [3], [6], [7] and a few of the results obtained were natural extensions of theorems concerning the complements and the lattices complemented in the classical sense. In the present paper we are going to continue the investigation of this sort.

For the concepts not defined here we refer to [1] or [5].

2. Projectivity of principal ideals

We recall the following definitions:

Definition 3. Two intervals $[a, b]$, $[c, d]$ of a lattice are said to be *transposed* if either the equations $a \cap d = c$ and $a \cup d = b$ or the equations $b \cap c = a$ and $b \cup c = d$ hold. In this case we write $[a, b] \sim [c, d]$.

Definition 4. Two intervals $[a, b]$, $[c, d]$ of a lattice L are said to be *projective* if there exists a finite sequence $[x_i, y_i]$ $(i = 0, 1, 2, \ldots, n)$ of intervals in L such that $[a, b] = [x_0, y_0]$, $[c, d] = [x_n, y_n]$ and $[x_{i-1}, y_{i-1}] \sim [x_i, y_i]$ for $i = 1, 2, \ldots, n$.

By a well-known theorem ([1], Theorem III. 19, p. 77), the principal ideals $[o, a]$ and $[o, b]$ of a complemented modular lattice L are projective if and only if there exists a finite sequence $c_0, c_1, \ldots, c_r$ of elements in L such that $a = c_0$, $b = c_r$ and any pair c_{j-1}, c_j $(j = 1, 2, \ldots, r)$ has a common complement. We are going to generalize this theorem for section complemented lattices.

First we prove the following lemma (conf. [7], Lemma):

Lemma 1. *Let L be a modular lattice with least element o and let p, q, v be elements of L such that p and q have a common (o, v)-complement c belonging to $[o, v]$.*

Then c is a common relative complement in $[o, v]$ of the elements $p_0 = p \cap v$ and $q_0 = q \cap v$.

Proof. Since $c \leqq v \leqq c \cup p$ by our assumptions, we get

$$c \cup p_0 = c \cup (p \cap v) = (c \cup p) \cap v = v$$

and, similarly, $c \cup q_0 = v$. Moreover, $c \cap p_0 = c \cap p \cap v = o$ and $c \cap q_0 = o$.

Now we formulate the main theorem of this paper:

Theorem 1. *Let L be a generalized complemented modular lattice with least element o. The principal ideals $[o, a]$ and $[o, b]$ of L are projective if and only if there exists an element $v \geqq a, b$ and a finite sequence $c_0, c_1, \ldots, c_r$ of elements in L such that $a = c_0, b = c_r$ and any pair c_{j-1}, c_j $(j = 1, 2, \ldots, r)$ has a common (o, v)-complement belonging to the interval $[o, v]$.*

We recall that a modular lattice is generalized complemented if and only if it is relatively complemented ([4], Theorems 2 and 3).

Proof. First we show that the condition given in the theorem is sufficient for the principal ideals $[o, a]$ and $[o, b]$ to be projective. In fact, this condition and Lemma 1 imply that the elements $d_j = c_j \cap v$ $(j = 0, 1, 2, \ldots, r)$ form a sequence such that $d_0 = a, d_r = b$ and any pair d_{j-1}, d_j $(j = 1, 2, \ldots, r)$ have a common relative complement r_j in $[o, v]$. Consequently,

$$[o, d_{j-1}] \sim [r_j, v] \sim [o, d_j] \quad (j = 1, 2, \ldots, r)$$

which verifies the projectivity of the principal ideals $[o, a]$ and $[o, b]$.

Next we show that the condition is necessary, too. Suppose that $[o, a]$ and $[o, b]$ are projective, i.e. that there exists a sequence $[x_i, y_i]$ $(i = 0, 1, 2, \ldots, n)$ with the properties described in Definition 4. Consider an element v such that $v \geqq y_i$ for all indices i. Then each interval $[x_i, y_i]$ is included by the sublattice $[o, v]$ which is complemented and modular. Thus the existence of a sequence $c_0, c_1, \ldots, c_r$ with properties given in the theorem follows at once from Theorem III. 19, cited above, of [1].

3. Congruence kernels

Let L be a complemented lattice and Θ a congruence relation on L. Let, further, K denote the kernel of Θ. Then, as well-known, K is an ideal of L and it has the following property:

(P_1) If the element a, b of L have a common complement, then $a \in K$ implies $b \in K$.

Under the additional condition of modularity, also the converse statement is true: If K is an ideal of a complemented modular lattice and K satisfies the condition (P_1), then there exists a congruence relation of L which has K as its kernel ([1], Theorem III. 20, p. 78). We generalize these results.

Theorem 2. *Let L be a lattice with least element o and Θ a congruence relation of L. Let, further, K denote the kernel of Θ. Then K has the following property:* (P_2) *If the elements a, b of L have a common $(o, a \cup b)$-complement, then $a \in K$ implies $b \in K$.*

Proof. Let a, b be any elements of L that have a common $(o, a\cup b)$-complement c. Then $a\cup c\geqq a\cup b\geqq b$. Thus $a\in K$ implies

$$b = b\cap(a\cup c) \equiv b\cap(o\cup c) = b\cap c = o(\Theta),$$

i.e. $b\in K$, indeed.

Theorem 3. *Let L be a generalized complemented modular lattice with least element o and let K be an ideal, satisfying (P_2), of L. Then K is the kernel of some congruence relation of L.*

Corollary. *In a generalized complemented modular lattice the congruence relations correspond one-to-one with the ideals satisfying (P_2).*

Proof. Let Θ be defined by $a_1\equiv a_2(\Theta)$ to mean that there exists an element t in K such that $a_1\cup t=a_2\cup t$. Then, obviously, Θ is a join-congruence (even without any assumption for the lattice). Thus we have only to show that $a_1\equiv a_2(\Theta)$ implies $a_1\cap u\equiv a_2\cap u(\Theta)$ for any elements a_1, a_2, u of L.

Suppose $a_1\equiv a_2(\Theta)$. Then, by the definition of Θ,

$$(1)\qquad a_1\cup t = a_2\cup t \quad \text{for some} \quad t\in K$$

and

$$(2)\qquad x\cup t \equiv x(\Theta) \quad \text{for each} \quad x\in L, \quad t\in L,$$

because $(x\cup t)\cup t=x\cup t$. It is sufficient to show that

$$(3)\qquad (a\cap u)\cup t \equiv (a\cup t)\cap(u\cup t) \quad (a = a_1, a_2),$$

because (1)—(3) imply

$$\begin{aligned} a_1\cap u &\equiv (a_1\cap u)\cup t \equiv (a_1\cup t)\cap(u\cup t) \equiv \\ &\equiv (a_2\cup t)\cap(u\cup t) \equiv (a_2\cap u)\cup t \equiv a_2\cap u(\Theta). \end{aligned}$$

In order to verify (3) consider the elements

$$\begin{aligned} p &= (a\cap u)\cup(u\cap t)\cup(t\cap a), \\ q &= (a\cup u)\cap(u\cup t)\cap(t\cup a), \\ e &= (u\cap t)\cup(a\cap(u\cup t)), \\ f &= (t\cap a)\cup(u\cap(t\cup a)), \\ g &= (a\cap u)\cup(t\cap(a\cup u)) = ((a\cap u)\cup t)\cap(a\cup u), \\ s &= (a\cup u)\cap t, \\ v &= (a\cap t)\cup(u\cap t) \end{aligned}$$

and introduce the brief notations

$$\begin{aligned} r &= (a\cap u)\cup t, \\ w &= (a\cup t)\cap(u\cup t) \end{aligned}$$

for the elements in (3). Then

$$[r, w] \sim [g, q] \sim [p, f] \sim [e, q] \sim [p, g] \sim [v, s], \tag{4}$$

because

$$r \cap q = r \cap ((a \cup u) \cap w) = r \cap (a \cup u) = g$$

and

$$r \cup q = r \cup ((a \cup u) \cap w) = (r \cup a \cup u) \cap w = w$$

by $r \leqq w$ and by the modularity,

$$g \cap f = f \cap e = e \cap g = p$$

and

$$g \cup f = f \cup e = e \cup g = q$$

by [1], Lemma II.3 (p. 38), finally

$$s \cup p = g \quad \text{and} \quad s \cap p = v$$

by the dual of the computation made for r and q. Since the lattice L is (generalized complemented and therefore, by Theorem 3 of [4], also) relatively complemented, there exists a relative complement r' of r in $[o, w]$ and a relative complement v' of v in $[o, s]$. With these relative complements we have

$$[o, w \cap r'] \sim [r, w] \quad \text{and} \quad [v, s] \sim [o, s \cap v']. \tag{5}$$

Relations (4) and (5) show that the principal ideals $[o, s \cap v']$ and $[o, w \cap r']$ are projective.

It follows, by Theorem 1 and by the first part of its proof that there can be given an element m and a finite sequence $d_0, d_1, \ldots, d_n$ in $[o, m]$ such that $d_0 = s \cap v'$, $d_n = w \cap r'$ and any pair d_{j-1}, d_j ($j = 1, 2, \ldots, n$) has a common (o, m)-complement d_j'. Since $d_{j-1} \cup d_j \leqq m$, the element d_j' is, a fortiori, a common $(o, d_{j-1} \cup d_j)$-complement of d_{j-1} and d_j. Moreover, by $s \cap v' \leqq s \leqq t$ and $t \in K$, the element $d_0 = s \cap v'$ belongs to the ideal K. It follows, by (P_2), that $d_1 \in K$ whence $d_2 \in K$ and so on; finally $w \cap r' = d_n \in K$. Hence, by the definition of Θ and by $r \leqq w$ we get

$$r \equiv r \cup (r' \cap w) = (r \cup r') \cap w = w(\Theta),$$

verifying (3). Thus the theorem is proved. The corollary is obvious.

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