The geometry of surfaces in parasymplectic spaces

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I. Parasymplectic space

1. *Definition of the parasymplectic space*

Let R^m denotes m-dimensional cartesian space and let Λ be an automorphism of R^m. We shall denote by this same symbol its matrix with respect to the canonical basis. Denote by $S_\Lambda(m)$ the subgroup of the group $GL(m)$ defined by:

$$S_\Lambda(m) = \{G \in GL(m) | G^T \Lambda G = \Lambda\}. \tag{1}$$

Consider a Klein space for which the basic space is R^m and its fundamental automorphisms group is the affine group reduced to the transformations $x \to Ax + a$ for which $A \in S_\Lambda(m)$. This space will be denoted by (R^m, Λ).

We shall deal with some special space, namely such one, for which Λ is antisymmetric and singular.

Definition 1. If Λ is antisymmetric and singular then we call the space (R^m, Λ) a parasymplectic one.

We have to find a fundamental differential invariant of second order and we investigate a differential geometry of surfaces in our space.

Theorem 1. *Let* $\Lambda^T \neq \Lambda$ *and* $\Lambda^T \neq -\Lambda$. *We put* $\Phi = \Lambda - \Lambda^T$. *Then* $S_\Lambda(m)$ *is a subgroup of* $S_\Phi(m)$.

PROOF. Let $G \in S_\Lambda(m)$, so we have

$$G^T \Lambda G = \Lambda$$

and then

$$G^T \Lambda^T G = \Lambda^T.$$

Hence

$$G^T \Phi G = \Phi.$$

The main consequence of this Theorem 1 is that each space (R^m, Λ) where $\Lambda^T \neq \Lambda$ and $\Lambda^T \neq -\Lambda$ is either a subspace of the symplectic space or of the parasymplectic space $(R^m, \Lambda - \Lambda^T)$.

2. *Length of a curve in parasymplectic space*

Let $t \to z(t)$ be a curve in the parasymplectic space (R^m, Λ).

Definition 2. The length of the curve $t \to z(t)$ contained between points $z(t_1)$, $z(t_2)$ is the number

$$\int_{t_1}^{t_2} (\dot z(t)^T \Lambda \ddot z(t))^{\frac{1}{3}}\, dt. \tag{2}$$

This kind of definition of length was first proposed by R. KRASNODĘBSKI [2]. It is easy to see that if we take any new parametrization $t \to s(t)$ then we have

$$(z \circ s)^{\cdot} = (\dot z \circ s)\dot s$$

$$(z \circ s)^{\cdot\cdot} = (\ddot z \circ s)\dot s^2 + (\dot z \circ s)\ddot s$$

$$(z \circ s)^{\cdot}(t)^T \Lambda (z \circ s)^{\cdot\cdot}(t) = \dot s(t)^3 \dot z(s(t))^T \Lambda \ddot z(s(t))$$

and this implies that the quantity (2) does not depends on parametrization.

Theorem 2. *The length of the curve defined by* (2) *is an invariant with respect to automorphisms of the parasymplectic space.*

PROOF. Let $t \to z(t)$ be a curve in (R^m, Λ). By any affine transformation it is mapped onto the curve

$$t \to (Az+a)(t) := Az(t) + a$$

where $A \in S_\Lambda(m)$.

Simple calculations show that

$$(Az+a)^{\cdot}(t)^T \Lambda (Az+a)^{\cdot\cdot}(t) = \dot z(t)^T \Lambda \ddot z(t)$$

what ends the proof.

Now are going to construct the geometry of the surfaces in parasymplectic space basing on the notion of introduced above length.

Contrary to the euclidean geometry this one is to be the geometry of second order due to the second order jets appering in the definition of the length.

II. Geometry of the surface in parasymplectic space

1. *Interior geometry*

Let M be a k-dimensional surface in (R^m, Λ). Suppose

$$u = (u^1, \dots, u^k) \to x(u) = \begin{bmatrix} x^1(u^1, \dots, u^k) \\ \vdots \\ x^m(u^1, \dots, u^k) \end{bmatrix}$$

be a local parametrization of a subdomine of M. Put

$$p_\alpha(u) = \begin{bmatrix} x^1_{|\alpha}(u) \\ \vdots \\ x^m_{|\alpha}(u) \end{bmatrix} \quad p_{\alpha\beta}(u) = \begin{bmatrix} x^1_{|\alpha\beta}(u) \\ \vdots \\ x^m_{|\alpha\beta}(u) \end{bmatrix} \quad \alpha, \beta = 1, \dots, k$$

where $x^i_{|\alpha}$ denotes the partial derivative of x^i with respect to α-coordinate. Obviously $p_\alpha(u), p_{\alpha\beta}(u)$ are the coordinates of the jet $j^2_u x$.

We introduce the invariants

$$P_{\alpha\beta}(u) = p_\beta(u)^T \Lambda p_\alpha(u) \tag{3}$$

$$Q_{\alpha\beta\gamma}(u) = p_\gamma(u)^T \Lambda p_{\alpha\beta}(u) \quad \alpha, \beta, \gamma = 1, \dots, k. \tag{4}$$

If we change the parametrisation

$$(u^{1'}, \dots, u^{k'}) \to \left(u^1(u^{1'}, \dots, u^{k'}), \dots, u^k(u^{1'}, \dots, u^{k'})\right)$$

the functions $P_{\alpha\beta}$ and $Q_{\alpha\beta\gamma}$ transforms as follow

$$P_{\alpha'\beta'} = A^\alpha_{\alpha'} A^\beta_{\beta'} P_{\alpha\beta} \tag{3'}$$

$$Q_{\alpha'\beta'\gamma'} = A^\alpha_{\alpha'} A^\beta_{\beta'} A^\gamma_{\gamma'} Q_{\alpha\beta\gamma} + A^\sigma_{\alpha'\beta'} A^\varrho_{\gamma'} P_{\sigma\varrho} \tag{4'}$$

where

$$A^\alpha_{\alpha'} = u^\alpha_{|\alpha'}, \quad A^\alpha_{\alpha'\beta'} = u^\alpha_{|\alpha'\beta'}.$$

Thus the functions $P_{\alpha\beta}$ are coordinates of some tensor field P on M.

Theorem 3. *The length of the curve* $t \to x(u(t))$ *on the surface M induced by* (2) *is given by the formula*

$$\int_{t_1}^{t_1} \left[P_{\beta\gamma}(u(t)) \ddot{u}^\beta(t) \dot{u}^\gamma(t) + Q_{\alpha\beta\gamma}(u(t)) \dot{u}^\alpha(t) \dot{u}^\beta(t) \dot{u}^\gamma(t)\right]^{\frac{1}{3}} dt. \tag{5}$$

PROOF. We have

$$\begin{aligned} (x \circ u)^\cdot(t) &= p_\alpha(u(t)) \dot{u}^\alpha(t) \\ (x \circ u)^{\cdot\cdot}(t) &= p_\alpha(u(t)) \ddot{u}^\alpha(t) + p_{\alpha\beta}(u(t)) \dot{u}^\alpha(t) \dot{u}^\beta(t) \end{aligned} \tag{6}$$

and using (3), (4), (6) we get

$$(x \circ u)^\cdot(t)^T \Lambda (x \circ u)^{\cdot\cdot}(t) = P_{\beta\gamma}(u(t)) \ddot{u}^\beta(t) \dot{u}^\gamma(t) + Q_{\alpha\beta\gamma}(u(t)) \dot{u}^\alpha(t) \dot{u}^\beta(t) \dot{u}^\gamma(t).$$

Comparing the last equality with (2) we obtain (5).

Definition 3. The tensor field P is said to be nonsingular iff

$$\det [P_{\alpha\beta}(u)] \neq 0. \tag{7}$$

Suppose now that M is a surface for which the tensor field P is nonsingular. The condition (7) in view of obvious

$$P_{\alpha\beta}(u) = -P_{\alpha\beta}(u) \tag{8}$$

implies that k must be an even integer.

Put

$$G^{\gamma}_{\alpha\beta}(u) = P^{\gamma\sigma}(u)Q_{\alpha\beta\sigma}(u) \tag{9}$$

where $[P^{\alpha\beta}]$ is uniquely determined from the equation

$$P_{\alpha\sigma}(u)P^{\beta\sigma}(u) = \delta^{\beta}_{\alpha}.$$

The condition (4′) implies that the functions $G^{\gamma}_{\alpha\beta}$ behave under transformations analogously as Christoffel's coefficients in the Riemann space. Moreover in view of (4)

$$G^{\gamma}_{\alpha\beta}(u) = G^{\gamma}_{\beta\alpha}(u). \tag{9′}$$

Definition 4. The pair $(P_{\alpha\beta}, Q_{\alpha\beta\gamma})$ will be called the metrical object of the surface M.

Denote ∇ the operator of covariant differentiation with respect $G^{\gamma}_{\alpha\beta}$.

Theorem 4.

$$\nabla_{\gamma}P_{\alpha\beta} = 0. \tag{10}$$

PROOF. Starting from (3), (4) we get

$$P_{\alpha\beta|\gamma}(u) = Q_{\alpha\gamma\beta}(u) - Q_{\beta\gamma\alpha}(u) \tag{11}$$

and then using (8), (9), (11)

$$\nabla_{\gamma}P_{\alpha\beta} = P_{\alpha\beta|\gamma} - P_{\alpha\sigma}G^{\sigma}_{\beta\gamma} - P_{\sigma\beta}G^{\sigma}_{\alpha\gamma} = \\ = Q_{\alpha\gamma\beta} - Q_{\beta\gamma\alpha} + Q_{\beta\gamma\alpha} - Q_{\alpha\gamma\beta} = 0.$$

This is a counterport of the Ricci Lemma in riemannian geometry.

Theorem 5.

$$P_{\alpha\beta|\gamma} + P_{\beta\gamma|\alpha} + P_{\gamma\alpha|\beta} = 0, \tag{12}$$

$$P^{\sigma\varrho}P_{\sigma\varrho|\alpha} = 2\sum_{\sigma} G^{\sigma}_{\alpha\sigma}. \tag{13}$$

The PROOF is a consequence of (11).

Definition 5. The curve in (R^m, Λ) is said to be isotropic iff the length of any arc contained in it is equal zero.

Definition 6. The geodesic curve $t \to x(u(t))$ on M is any curve satisfying.

$$\ddot{u}^{\gamma}(t) + G^{\gamma}_{\alpha\beta}(u(t))\dot{u}^{\alpha}(t)\dot{u}^{\beta}(t) = \lambda(t)\dot{u}^{\gamma}(t). \tag{14}$$

Theorem 6. *Each geodesic curve is isotropic.*

PROOF. Looking on (8), (9), (14) we get

$$P_{\beta\gamma}\ddot{u}^{\beta}\dot{u}^{\gamma} + Q_{\alpha\beta\gamma}\dot{u}^{\alpha}\dot{u}^{\beta}\dot{u}^{\gamma} = P_{\sigma\gamma}(\ddot{u}^{\sigma} + G^{\sigma}_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta})\dot{u}^{\gamma} = 0$$

what together with (5) ends the proof.

2. *Exterior geometry of hypersurfaces*

Consider the parasymplectic space (R^m, Λ) of odd dimension m with rank $\Lambda = m-1$.

In view of singularity of the matrix Λ there exists a non-zero vector $n \in R^m$ such that

$$\Lambda n = 0. \tag{15}$$

Consider the constant vector field $x \to N(x)$ on M obtained from the vector n by translation to the point $x \in M$.

Theorem 7. *If* rank $\Lambda = m-1$ *then the conditions*

1° *the tensor field P is nonsingular,*

2° *the vectors $p_1(u), \ldots, p_{m-1}(u)$, $N(u)$ are linearly independent,*

are equivalent.

PROOF. Consider the matrix $[p_1(u), \ldots, p_{m-1}(u), N(u)]$ consisting of the column vectors $p_1(u), \ldots, p_{m-1}(u), N(u)$. Notice that

$$[p_1(u), \ldots, p_{m-1}(u),\ N(u)]^T[-\Lambda p_1(u), \ldots, -\Lambda p_{m-1}(u),\ N(u)] =$$

$$= \left[\begin{array}{c:c} P_{\alpha\beta}(u) & * \\ \hdashline 0 & \sum_{i=1}^{m} (n^i)^2 \end{array}\right].$$

The nonsingularity of P implies 2°.

If the vectors $p_1(u), \ldots, p_{m-1}(u)$, $N(u)$ are linearly independent then the condition rank $\Lambda = m-1$ implies the linear independence of vectors $-\Lambda p_1(u), \ldots, -\Lambda p_{m-1}(u), N(u)$. Thus $2° \Rightarrow 1°$.

Restrict the vector field p_α to the field of β's parametric line. We can differentiate along this line in R^m obtaining the vector field $p_{\alpha\beta}$.

Hence by using Theorem 7 we get the following "Gauss decomposition".

$$p_{\alpha\beta}(u) = \Gamma^{\sigma}_{\alpha\beta}(u)\, p_\sigma(u) + B_{\alpha\beta}(u)\, N(u) \tag{16}$$

so for the vector field N "the Weingarten's formulas" are trivial

$$N_{|\alpha}(u) = 0. \tag{17}$$

Theorem 8.

$$\Gamma^{\gamma}_{\alpha\beta}(u) = G^{\gamma}_{\alpha\beta}(u).$$

PROOF. Composing (16) and (4) we get

$$Q_{\alpha\beta\gamma}(u) = p_\gamma(u)^T \Lambda\big(\Gamma^{\sigma}_{\alpha\beta}(u)\, p_\sigma(u) + B_{\alpha\beta}(u)\, N(u)\big) = \Gamma^{\sigma}_{\alpha\beta}(u)\, P_{\sigma\gamma}(u),$$

what implies (15) completing the proof.

Theorem 9.

$$B_{\alpha\beta}(u) = \frac{\det[p_1(u) \ldots p_{m-1}(u)\, p_{\alpha\beta}(u)]}{\det[p_1(u) \ldots p_{m-1}(u) N(u)]}. \tag{19}$$

PROOF. From the decomposition (16) we have

$$\det[p_1(u)\dots p_{m-1}(u)\,p_{\alpha\beta}(u)] = \det[p_1(u)\dots p_{m-1}(u)\,B_{\alpha\beta}(u)\,N] =$$
$$= B_{\alpha\beta}(u)\det[p_1(u)\dots p_{m-1}(u)\,N(u)].$$

Hence we obtain (19).

It is easy to see that the functions $B_{\alpha\beta}$ are coordinates of some tensor field B on M and

$$B_{\alpha\beta}(u) = B_{\alpha\beta}(u). \tag{20}$$

Definition 7. The tensor field B we will called the imbedding tensor of the surface M.

Let us look now for some counterports of Codazzi's formulas and count down the coordinates of the curvature tensor.

Theorem 10. *The following formulas hold*

$$\nabla_\gamma B_{\alpha\beta} = \nabla_\beta B_{\alpha\gamma} \tag{21}$$

$$R^\delta_{\alpha\beta\gamma} = 0. \tag{22}$$

PROOF. From Gauss formula (16) we derive

$$p_{\alpha\beta|\gamma} = (G^\sigma_{\alpha\beta|\gamma} + G^\delta_{\alpha\beta}G^\sigma_{\delta\gamma})\,p_\sigma + (B_{\alpha\beta|\gamma} - G^\delta_{\alpha\beta}B_{\delta\gamma})\,N.$$

Hence

$$0 = p_{\alpha\beta|\gamma} - p_{\alpha\gamma|\beta} = R^\sigma_{\gamma\alpha\beta}\,p_\sigma + (\nabla_\gamma B_{\alpha\beta} - \nabla_\beta B_{\alpha\gamma})\,N$$

and the proof easily follows.

Theorem 11. *Let $D\subset R^{2m}$ be an open and connected domain of three matrices fields $[P_{\alpha\beta}]$, $[Q_{\alpha\beta\gamma}]$, $[B_{\alpha\beta}]$, $\alpha, \beta, \gamma = 1, \dots, 2m$ such that*

$$P_{\alpha\beta} = -P_{\beta\alpha}, \quad \det[P_{\alpha\beta}] \neq 0$$

$$Q_{\alpha\beta\gamma} = Q_{\beta\alpha\gamma}, \quad B_{\alpha\beta} = B_{\beta\alpha}$$

$$P_{\alpha\beta|\gamma} + Q_{\beta\gamma\alpha} - Q_{\alpha\gamma\beta} = 0 \tag{P}$$

$$B_{\alpha\beta|\gamma} - B_{\alpha\gamma|\beta} + G^\sigma_{\alpha\gamma}B_{\sigma\beta} - G^\sigma_{\alpha\beta}B_{\sigma\gamma} = 0 \tag{B}$$

$$G^\delta_{\alpha\beta|\gamma} - G^\delta_{\alpha\gamma|\beta} + G^\sigma_{\alpha\beta}G^\delta_{\sigma\gamma} - G^\sigma_{\alpha\gamma}G^\delta_{\sigma\beta} = 0 \tag{R}$$

where $G^\gamma_{\alpha\beta} = P^{\gamma\sigma}Q_{\alpha\beta\sigma}$.

Then for any point $u\in D$ there exists a neighbourhood U of u and a diffeomorphism x of U into a some parasymplectic space (R^{2m+1}, Λ) with the following properties:

The pair $(P_{\alpha\beta}, Q_{\alpha\beta\gamma})$ is the metrical object of the surface M parametrized by x; moreover the function $B_{\alpha\beta}$ are the coordinates of the imbedding tensor of M.

PROOF. Consider the Pfaff's system

$$\begin{cases} dx^i = x^i_{|\alpha}\,du^\alpha \\ dx^i_{|\alpha} = G^\gamma_{\alpha\beta}x^i_{|\gamma}\,du^\beta + B_{\alpha\beta}N^i\,du^\beta \\ dN^i = 0. \end{cases} \tag{a}$$

The conditions $B_{\alpha\beta}=B_{\beta\alpha}$, $G^{\gamma}_{\alpha\beta}=G^{\gamma}_{\beta\alpha}$, (B) i (R) implies the integrability of our system.

Let $u_0\in D$ and let the vectors $a_1, \dots, a_{2m}$ in R^{2m+1} be linearly independent. We define the vectors $a_{\alpha\beta}$ by the formula

(1°) $$A_\alpha = AX^{-1}Y_\alpha$$

where

$$A = [a_1 \dots a_{2m}] \quad \text{(rectangle matrix)}$$

$$X = [P_{\alpha\beta}(u_0)] \quad \text{(nonsingular matrix)}$$

$$Y_\alpha = [Q_{\alpha\beta\gamma}(u_0)]_{\beta,\gamma} \quad \text{(a sequence of square matrices)}$$

$$A_\alpha = [a_{\alpha 1} \dots a_{\alpha 2m}] \quad \text{(a sequence of rectangle matrices).}$$

We find some antisymmetric matrix Λ of degree $2m+1$ such that rank $\Lambda=2m$, $P_{\alpha\beta}(u_0)=a^T_\beta \Lambda a_\alpha$, $Q_{\alpha\beta\gamma}(u_0)=a^T_\gamma \Lambda a_{\alpha\beta}$, i.e.

(2°) $$\operatorname{rank} \Lambda = 2m$$

(3°) $$X = A^T \Lambda A$$

(4°) $$Y_\alpha = A^T \Lambda A_\alpha.$$

We note that (1°) and (3°) implies (4°). Since rank $A=$rank $X=2m$ so from (3°) we obtain (2°). Thus (3°) defined some matrix Λ. Let $\hat{N}$ be defined by the condition $\Lambda\hat{N}=0$.

Let x be a solution of the system (a) with the initial condition

$$x(u_0) = a, \quad x_{|\alpha}(u_0) = a_\alpha, \quad N(u_0) = \hat{N}.$$

Put

$$\hat{P}_{\alpha\beta}(u) = x_{|\beta}(u)^T \Lambda x_{|\alpha}(u)$$

$$\hat{Q}_{\alpha\beta\gamma}(u) = x_{|\gamma}(u)^T \Lambda x_{|\alpha\beta}(u).$$

Of course, we have

(b) $$\begin{cases} \hat{P}_{\alpha\beta}(u_0) = P_{\alpha\beta}(u_0) \\ \hat{Q}_{\alpha\beta\gamma}(u_0) = Q_{\alpha\beta\gamma}(u_0). \end{cases}$$

Now we put

$$\hat{G}^{\gamma}_{\alpha\beta}(u) = \hat{P}^{\gamma\sigma}(u) Q_{\alpha\beta\sigma}(u),$$

the functions $\hat{G}^{\gamma}_{\alpha\beta}$ are defined in some neighbourhood of the point u_0.

We find the differential equations for the functions $\hat{P}_{\alpha\beta}$ and $\hat{Q}_{\alpha\beta\gamma}$.

$$\hat{P}_{\alpha\beta|\gamma} = x^T_{|\beta\gamma}\Lambda x_{|\alpha} + x^T_{|\beta}\Lambda x_{|\alpha\gamma} = \hat{G}^{\sigma}_{\beta\gamma}\hat{P}_{\alpha\sigma} + \hat{G}^{\sigma}_{\alpha\gamma}\hat{P}_{\sigma\beta} = \hat{Q}_{\alpha\gamma\beta} - \hat{Q}_{\beta\gamma\alpha}$$

$$\hat{Q}_{\alpha\beta\gamma|\delta} = x^T_{|\gamma\delta}\Lambda x_{|\alpha\beta} + x^T_{|\gamma}\Lambda x_{|\alpha\beta\delta} = \hat{G}^{\sigma}_{\gamma\delta}\hat{G}^{\varrho}_{\alpha\beta}\hat{P}_{\varrho\sigma} + (\hat{G}^{\sigma}_{\alpha\beta|\delta} + \hat{G}^{\varrho}_{\alpha\beta}\hat{G}^{\sigma}_{\varrho\delta})\hat{P}_{\sigma\gamma}.$$

Thus we obtain the following system of differential equations

(c) $$\begin{cases} \hat{P}_{\alpha\beta|\gamma} = \hat{Q}_{\alpha\gamma\beta} - \hat{Q}_{\beta\gamma\alpha} \\ \hat{Q}_{\alpha\beta\gamma|\delta} = \hat{G}^{\sigma}_{\gamma\delta}\hat{G}^{\varrho}_{\alpha\beta}\hat{P}_{\varrho\sigma} + (\hat{G}^{\sigma}_{\alpha\beta|\delta} + \hat{G}^{\varrho}_{\alpha\beta}\hat{G}^{\sigma}_{\varrho\delta})\hat{P}_{\sigma\gamma}. \end{cases}$$

We show that

$$(Q)\qquad Q_{\alpha\beta\gamma|\lambda} = P_{\sigma\gamma}(G^{\sigma}_{\alpha\beta|\lambda} + G^{\sigma}_{\varrho\gamma}G_{\alpha\beta}) + G^{\varrho}_{\gamma\lambda}Q_{\alpha\beta\varrho}.$$

The condition

$$P_{\alpha\varphi}P^{\beta\varphi} = \delta^{\beta}_{\alpha}$$

and (P) implies that

$$P^{\beta\lambda}{}_{|\gamma} = -P^{\sigma\lambda}P_{\sigma\varrho|\gamma}P^{\beta\varrho} = -P^{\sigma\lambda}(Q_{\sigma\gamma\varrho} - Q_{\varrho\gamma\sigma})P^{\beta\varrho} = -P^{\sigma\lambda}G^{\beta}_{\sigma\gamma} - P^{\beta\varrho}G^{\lambda}_{\varrho\gamma}.$$

Hence

$$G^{\delta}_{\alpha\beta|\gamma} = P^{\delta\varrho}{}_{|\lambda}Q_{\alpha\beta\varrho} + P^{\delta\varrho}Q_{\alpha\beta\varrho|\lambda} = (-G^{\delta}_{\sigma\lambda}P^{\sigma\varrho} - G^{\varrho}_{\sigma\lambda}P^{\delta\sigma})Q_{\alpha\beta\varrho} + P^{\delta\varrho}Q_{\alpha\beta\varrho|\lambda} =$$
$$= -G^{\delta}_{\sigma\lambda}G^{\sigma}_{\alpha\beta} - G^{\varrho}_{\sigma\lambda}P^{\delta\sigma}Q_{\alpha\beta\varrho} + P^{\delta\varrho}Q_{\alpha\beta\varrho|\lambda}.$$

Multiplying by $P_{\delta\gamma}$ we obtain (Q). The functions $P_{\alpha\beta}$ and $Q_{\alpha\beta\gamma}$ with respect to (P) and (Q) statisfying the system (c). The initial conditions (b) implies

$$\hat{P}_{\alpha\beta} \equiv P_{\alpha\beta} \quad \text{and} \quad \hat{Q}_{\alpha\beta\gamma} \equiv Q_{\alpha\beta\gamma}.$$

Moreover B are coordinates of the imbedding tensor of M.

3. *Exterior geometry of surfaces in parasymplectic space* (R^3, Λ)

S. Watanabe [4, 5] investigated the geometry of surfaces in the affine space K^3. Our aim in this chapter is to prove the following

Theorem 12.

$$K^3 = \left(R^3, \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\right).$$

Proof. Before starting the proof let us recall the definition of K^3. To do so it is enough to show the centroaffine group which for K^3 is

$$(24)\qquad \left\{A \in \mathrm{GL}(3) \,\middle|\, \sum_{i=1}^{3} a^j_i = \det A \ \text{ for } j=1, 2, 3\right\}.$$

It is easy to see that if

$$(25)\qquad a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

then this group can be defined also as

$$(24')\qquad \{A \in \mathrm{GL}(3) \ \ Aa = \det A \cdot a\}$$

as it was done in [1].

Suppose

$$\Lambda = \begin{bmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & \varkappa \\ -\mu & -\varkappa & 0 \end{bmatrix} \quad n = \begin{bmatrix} \varkappa \\ -\mu \\ \lambda \end{bmatrix}. \tag{26}$$

Theorem 13. *For any* $G \in GL(3)$ *the conditions*

(a) $$G^T \Lambda G = \Lambda$$

(b) $$Gn = \det G \cdot n$$

are equivalent.

PROOF. Suppose

$$G = \begin{bmatrix} a & b & c \\ u & v & w \\ p & q & r \end{bmatrix} \in S_\Lambda(3).$$

then the conditions $G^T \Lambda G = \Lambda$ leads to the equations

$$(-\lambda u - \mu p)\, b + (\lambda a - \varkappa p)\, v + (\mu a + \varkappa u)\, q = \lambda$$

$$(-\lambda v - \mu q)\, c + (\lambda a - \varkappa p)\, w + (\mu a + \varkappa u)\, r = \mu$$

$$(-\lambda v - \mu q)\, c + (\lambda b - \varkappa q)\, w + (\mu b + \varkappa v)\, r = \varkappa$$

or in an equivalent way to

$$\varkappa(uq - pv) = \lambda + \lambda ub + \mu bp - \lambda av - \mu aq$$

$$\varkappa(ur - pw) = \mu + \lambda uc + \mu bc - \lambda aw - \mu ar$$

$$\varkappa(vr - qw) = \varkappa + \lambda vc + \mu qc - \lambda bw - \mu br$$

and counting down $\varkappa \det G$, $-\mu \det G$ and $\lambda \det G$ we get

$$\varkappa \det G = a\varkappa(vr - wq) - b\varkappa(ur - wp) + c\varkappa(uq - vp) = a\varkappa - b\mu + c\lambda,$$

$$-\mu \det G = u\varkappa - v\mu + w\lambda,$$

$$\lambda \det G = p\varkappa - q\mu + r\lambda.$$

On the other hand

$$Gn = \begin{bmatrix} a & b & c \\ u & v & w \\ p & q & r \end{bmatrix} \begin{bmatrix} \varkappa \\ -\mu \\ \lambda \end{bmatrix} = \begin{bmatrix} a\varkappa - b\mu + c\lambda \\ u\varkappa - v\mu + w\lambda \\ p\varkappa - q\mu + r\lambda \end{bmatrix}$$

so $Gn = \det G \cdot n$ implying (a)$\Rightarrow$(b).

New the condition (b) may be written as

$$n = \det G \cdot G^{-1} n,$$

hence

$$\varkappa = (vr-qw)\varkappa+(br-cq)\mu+(bw-vc)\lambda,$$

$$\mu = (ur-wp)\varkappa+(ar-cp)\mu+(aw-cu)\lambda,$$

$$\lambda = (uq-vp)\varkappa+(aq-bp)\mu+(av-ub)\lambda$$

which is exactly what we need to show that (b)⇒(a).

Let pass now to the proof of Theorem 12.

PROOF. Putting in Theorem 13 $n=a$ and using the definition (24′) we get the conclusion of Theorem 12 immediately.

4. *Exterior geometry of k-dimensional surfaces in* $(R^m, \ )$

Our aim now is to investigate k-dimensional surfaces M which are subject to

$$(*) \qquad \det [P_{\alpha\beta}(u)] \neq 0$$

$$\binom{*}{*} \qquad k = \operatorname{rank} \Lambda.$$

These two conditions imply that $\dim \operatorname{Ker} \Lambda = m-k$ where k is an even integer.

Let the vectors $n_1, \ldots, n_{m-k}$ constitute the bases in Ker Λ. Let us define a constant vector field N_i on M translating vectors n_i to the point $x(u) \in M$.

Theorem 14. *If* $\operatorname{rank} \Lambda = k$ *then the conditions:*

(a) *the tensor field P is nonsingular,*

(b) *the vectors* $p_1(u), \ldots, p_k(u), N_1(u), \ldots, N_{m-k}(u)$ *are linearly independent are equivalent.*

PROOF. We have

$$[p_1(u) \ldots p_k(u)\, N_1(u) \ldots N_{m-k}(u)]^T [-\Lambda p_1(u) \ldots -\Lambda p_k(u)\, N_1(u) \ldots N_{m-k}(u)] =$$

$$= \left[\begin{array}{c:c} [P_{\alpha\beta}(u)]_{\alpha,\beta=1,\ldots,k} & * \\ \hdashline 0 & \left[\sum_{r=1}^{m} n_i^r n_j^r\right]_{i,j=1,\ldots,m-k} \end{array}\right].$$

In view of the independence of $N_1, \ldots, N_{m-k}$ and the generalized Lagrange's theorem on determinants, the matrix

$$\left[\sum_{r=1}^{m} n_i^r n_j^r\right]_{i,j=1,\ldots,m-k}$$

is nonsingular so (a)⇒(b).

If the vectors $p_1(u), \ldots, p_k(u), N_1(u), \ldots, N_{m-k}(u)$ are linearly independent then the condition $\operatorname{rank} \Lambda = k$ imply that the vectors $-\Lambda p_1(u), \ldots, -\Lambda p_k(u), N_1(u), \ldots, N_{m-k}(u)$ are also linearly independent. Thus (b)⇒(a).

The consequence of this is the following formula

$$p_{\alpha\beta}(u) = \tilde{\Gamma}^{\sigma}_{\alpha\beta}(u)\, p_{\sigma}(u) + B^{i}_{\alpha\beta}(u)\, N_i(u).$$

Analogously as in chapter 2 we can show

Theorem 15.

(27) $$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = G^{\gamma}_{\alpha\beta}$$

(28) $$R^{\delta}_{\alpha\beta\gamma} = 0$$

(29) $$\nabla_{\gamma} B^{i}_{\alpha\beta} = \nabla_{\beta} B^{i}_{\alpha\gamma}$$

(30) $$B^{i}_{\alpha\beta} = \frac{\det[p_1, \ldots, p_k, N_1 \ldots N_{i-1}, p_{\alpha\beta}, N_{i+1} \ldots N_{m-k}]}{\det[p_1 \ldots p_k, N_1 \ldots N_{m-k}]}.$$

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