Radicals of the semiring of abelian groups

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i. The isomorhism classes of abelain groups under the operations direct sum, and tensor product, from a semiring which is a class. In section *iii*, several radicals of this semiring will be Facts. concerning computed. semirings in general, which are employed in *iii*, will be introduced in section *ii*.

A group will always be meant to be an abelian group.

ii. Let S be a semiring with $0\in S$ satisfying $a+0=a$, and $a\cdot 0=0\cdot a=0$ for all $a\in S$.

Definition 2.1. A subset $I\subseteqq S$ is said to be an ideal in S if:

1) $a, b\in I\Rightarrow a+b\in I$,

2) $a\in I$, $x\in S\Rightarrow ax\in I$, and $xa\in I$, and

3) $0\in I$.

If in addition for all $a\in I$, and $x\in S$, $a+x\in I$ implies that $x\in I$, then I is said to be a k-ideal.

Undefined terms used here are meant to have the same meaning as their counterparts in ring theory.

Definition 2.2. An ideal I in S is said to be modular if there is an $e\in S$ such that for each $x\in S$, there exists an $i(x)\in I$ with $ex=x+i(x)$.

Clearly, every ideal in a semiring with identity is modular.

Notation 2.3. The sum of all nilpotent ideals in S will be denoted $N_0(S)$. For every ordinal τ, inductively define $N_{\tau+1}(S)$ to be the ideal in S such that $N_{\tau+1}(S)/N_\tau(S)$ is the sum of all the nilpotent ideals in $S/N_\tau(S)$. For τ a limit ordinal, define $N_\tau(S)=\bigcup_{\gamma<\tau} N(S)$.

For a definition of S/I, I an ideal in S, see [3, p. 164].

Definition 2.4. The upper nil radical of S, $UN(S)$, is the sum of all the nil ideals in S.

Definition 2.5. The Levitzki radical of S, $L(S)$, is the sum of all the locally nilpotent ideals in S [1, p. 262; 8, p. 91].

Definition 2.6. The prime radical of S, $P(S)=\cap\{P\,|\,P$ is a prime ideal in $S\}$.

Definition 2.7. The Jacobson radical of $S, J(S)=\cap\{M|M$ is a modular maximal ideal in $S\}$.

Lemma 2.8. *Let S be a semiring with identity, and let K be the set of non-units in S. S has a unique maximal ideal M iff K is an ideal, in which case $M=K$.*

PROOF. As in ring theory [2, theorem 3-3].

iii. Let Ab be the class of abelian groups. For every $A\in Ab$, let $[A]$ be the isomorphism class of A. Write $[Ab]=\{[A]|A\in Ab\}$. For $A, B\in Ab$, define $[A]+[B]=[A+B]$, and $[A]\cdot[B]=[A\otimes B]$. These two operations clearly make $[Ab]$ a commutative semiring, with identity $[Z]$, Z the group of integers.

Notation 3.1. Let A be a torsion free abelian groups. The rank of A will be denoted by $r(A)$. If $r(A)=1$, then $T(A)$ will signify the type of A [4, vol. 2, p. 109].

Notation 3.2. For all $A\in Ab$, the torsion part of A will be denoted by A_t. For every prime p, A_p will signify the p-component of A_t.

Notation 3.3. $T=\{[A]|A$ is a torsion group$\}$.

Notation 3.4. $DT=\{[A]|A$ is a divisible torsion group$\}$.

Notation 3.5. For every prime p, write $D_p=\{[A]|A_t$ is a p-divisible group$\}$.

Notation 3.6. $M=\{[A]|A\not\cong Z\}$.

Lemma 3.7. *Let P be a prime ideal in $[Ab]$, and let A be a torsion group. If $[A]\notin P$, then $[D]\in P$ for every divisible group D.*

PROOF. $D\otimes A=0$ for every divisible group D.

Similarly we have:

Lemma 3.8. *Let P be a prime ideal in $[Ab]$, and let D be a divisible group. If $[D]\notin P$, then $[T]\in P$ for every torsion group A.*

Corollary 3.9. *$DT\subseteqq P$ for every prime ideal P in $[Ab]$.*

Lemma 3.10. *T is a prime k-ideal in $[Ab]$.*

PROOF. T is clearly a k-ideal in $[Ab]$.

Let $A, B\in Ab$, and suppose that neither A nor B is a torsion group. Let A' and B' be nonzero, torsion free subgroups of A and B respectively. $A'\otimes B'$ is a nonzero torsion free group, and is isomorphic to a subgroup of $A\otimes B$ [5, theorem 2]. Hence $[A]\cdot[B]\notin T$.

Lemma 3.11. *For every prime number p, D_p is a prime k-ideal in $[Ab]$.*

PROOF. D_p is clearly a k-ideal in $[Ab]$. Let $A, B\in Ab$, and suppose that $(A\otimes B)_t$ is p-divisible, $A_t\otimes B_t$ is isomorphic to a direct summand of $(A\otimes B)_t$ [4, theorm 61.5], and hence must be p-divisible. $A_p\otimes B_p$ is a direct summand of $A_t\otimes B_t$, and is therefore p-divisible. However, $A_p\otimes B_p$ is a direct sum of cyclic p-groups, [4, theorem 61.3]. Therefore $A_p\otimes B_p=0$. Let P_A and P_B be p-basic subgroups of A_p

and B_p respectively. $0=A_p\otimes B_p\cong P_A\otimes P_B$ [4, theorem 61.1]. Since P_A and P_B are both direct sums of cyclic p-groups, $P_A\otimes P_B=0$ implies that either $P_A=0$, or $P_B=0$. This in turn yields that either A_t or B_t is p-divisible.

Theorem 3.12. $P([Ab])=DT$.

PROOF. $DT\subseteq P(Ab)$ by corollary 3.9. Lemmas 3.10 and 3.11 yield that

$$P([Ab])\subseteq T\cap\{\bigcap_{p \text{ a prime}} D_p\}=DT.$$

Theorem 3.13. *For every ordinal* τ,

$$N_\tau([Ab])=L([Ab])=UN([Ab])=DT.$$

PROOF. Clearly $N_0([Ab])\subseteq L([Ab])\subseteq UN([Ab])$, and $N_0([Ab])\subseteq N_\tau([Ab])\subseteq \subseteq UN([Ab])$ for every ordinal τ. It therefore suffices to show that $UN([Ab])=DT$, and that $DT\subseteq N_0([Ab])$.

1) $UN([Ab])=DT$: Let I be a nil ideal in $[Ab]$, and let $[A]\in[Ab]$. There exists a positive integer n such that $\tilde{A}=\underbrace{A\otimes\ldots\otimes A}_{n\text{-times}}=0$.

Let A' be a torsion free subgroup of A. Put $\tilde{A}'=\underbrace{A'\otimes\ldots\otimes A'}_{n\text{-times}}$.

The sequence $0\to\tilde{A}'\to\tilde{A}$ is exact. If $A'\neq0$, then $\tilde{A}'\neq0$. Therefore A must be a torsion group. If A is not divisible, then A possesses a cyclic, direct summand $B\neq0$. Write $\tilde{B}=\underbrace{B\otimes\ldots\otimes B}_{n\text{-times}}$. Clearly $\tilde{B}\neq0$, and $\tilde{B}$ is a direct summand of $\tilde{A}$. A contradiction. Hence A is divisible, and $I\subseteq DT$. This yields that $UN([Ab])\subseteq DT$.

Since $(DT)^2=0$, clearly $DT\subseteq UN([Ab])$.

2) $DT\subseteq N_0([Ab])$: This follows immediately from the fact that $(DT)^2=0$.

Theorem 3.14 *M is the unique maximal ideal in* $[Ab]$.

PROOF. By lemma 2.8 it suffices to show that every $[A]\in M$ is a non-unit in $[Ab]$, or that for $A, B\in Ab$, $A\otimes B\cong Z$ implies that $A\cong B\cong Z$.

Suppose that $A\otimes B\cong Z$.

1) If either A or B is a torsion group, then so is $A\otimes B$.

2) If A and B are both torsion free, then $r(A)\cdot r(B)=r(A\otimes B)=1$. Therefore $r(A)=r(B)=1$. However, $(0, \ldots, 0, \ldots)=T(A\otimes B)\cong T(A)+T(B)$. Therefore $T(A)= =T(B)=(0, \ldots, 0$, and), and $A\cong B\cong Z$.

3) Suppose that A is a mixed group. $Z\cong A\otimes B\cong(A\otimes B)/(A\otimes B)_t\cong A/A_t\otimes B/B_t$ [4, theorem 61.5]. By 2) we have that $A/A_t\cong B/B_t\cong Z$. Therefore $A_t\cong A_t\otimes B/B_t$. However, $A_t\otimes B/B_t$ is a direct summand of $(A\otimes B)_t$ [4, theorem 61.5]. A contradiction.

Corollary 3.15. $I([Ab])=M$.

iv. Results concerning the structure of the tensor product of groups, which may be found in [4, Section 61] were employed in section *iii*. Several of these results were shown to remain true for the tensor product of modules over a Dedekind ring in [5, and 6]. The generalization of a few more of the results in [4, section 61] to modules over a Dedekind ring, would yield that all the results obtained here remain true for the semiring of isomorphism classes of modules over a Dedekind ring.

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