A product theorem for a paracompact-like family of properties

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Abstract. In this paper we introduce a way to associate a topological property to each property of a family of subsets of an space in such a way that the correspondent property to local finiteness is just paracompactness.

We study the productivity of those properties under a compact factor, generalizing the study done in [1] for products of C-spaces, and we also study how they inherit to closed sets.

1. Introduction

We begin with the definition that motivated the paper.

Definition 1. Let P be a topological property satisfied by a family of subsets of a topological space and preserved by subfamilies (i.e., if $\mathcal{U}$ has P and $\mathcal{V}$ is a subfamily of $\mathcal{U}$, $\mathcal{V}$ also has P).

A topological space X is a P^*-space if and only if for any sequence of open covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X, there exists a sequence of families of open sets $\mathcal{V}_n$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a refinement of $\mathcal{U}_n$, $\mathcal{V}_n$ has property P and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ covers X.

As examples of such properties of families of sets one can consider the local finiteness, the point finiteness, the finiteness and so on. The example that motivated this paper is the property pairwise disjoint. For such property, P^*-spaces are called C-spaces in [1].

The following result shows the status of P^* among other properties.

Theorem 1.1. *Let X be a topological space.*

1. *If each open cover of X has a open refinement that is a cover and has property P, then X is P^*.*

2. *If X is P^*, then each open cover of X has a open refinement that is a cover and can be written as the countable union of families that have property P.*

PROOF. 1. Apply the hypothesis to each covering of the sequence.

2. Given a covering $\mathcal{U}$, apply the hypothesis to the sequence $\mathcal{U}_n = \mathcal{U}$ and let $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$. $\qquad\square$

Since when $P =$ finite, the first part of 1 of Theorem 1.1 is compactness, P^* is called Hurewicz in 3.2.16 of [2] and the second part of 2 of Theorem 1.1 is Lindelöf, the three assertions are not equivalent in general.

However, when P is local finiteness, a famous theorem of MICHAEL shows that the first part of 1 and the second part of 2 of Theorem 1.1 are equivalent (see 20.7 of [3]), so P^* is just paracompactness.

The usual way to construct new generalizations of paracompactness was replacing $P =$ local finiteness with another suitable property (when one take point finite obtains metacompact, and so on). The above observation leads to another way to obtain generalizations of paracompactness: consider P^* for suitable properties. So Definition 1 can be considered as a collection of definitions of paracompact-like properties that may share common properties.

Sometimes this procedure will reduce to known definitions as in the case of local finiteness and paracompactness; sometimes will not (for example, pairwise disjointness and C-spaces; finiteness and Hurewicz spaces).

2. Product theorem. Closed subspaces.

In this section we are going to study how is the behavior of P^* under products. Since it is known that the product of two paracompact spaces may be non-paracompact, and the best known product theorem for paracompactness is that the product of a compact space and a paracompact space is paracompact and we also know that the product of a compact C-space with a C-space is a C-space (see [1]), one can easily guess that the natural product theorem for P^* needs a compactness hypothesis over one of the factors and some condition about the productivity of P like the following.

Definition 2. Let P be a topological property satisfied by a family of subsets of a topological space. We say that P is productive if whenever $\mathcal{U}$ is a family of subsets of a set X having property P and for each $\mathcal{U}$ there is a family $\mathcal{V}(\mathcal{U})$ of subsets of a set Y having property P, the family $\{U \times V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}(\mathcal{U})\}$ has property P.

Theorem 2.1. *Let P be a productive property of a family of subsets of a set. The product of a P^*-compact with a P^*-space is a P^*-space.*

PROOF. Let X and Y be P^*-spaces, with Y compact, and let us show that the product $X \times Y$ is P^*.

Let a countable collection of open covers of $X \times Y$ be given. We rewrite this collection as a sequence of such countable collections

$$\{\{\mathcal{U}_{m,n} : n \in \mathbb{N}\} : m \in \mathbb{N}\}$$

where we may assume that each open cover has the form $\mathcal{U}_{m,n} = \{A^i_{m,n} \times B^i_{m,n} : i \in I_{m,n}\}$, where each $A^i_{m,n}$ is open in X and each $B^i_{m,n}$ is open in Y.

Fix $m \in \mathbb{N}$ and let $x \in X$ be fixed but arbitrary. For each $n \in \mathbb{N}$, we use the compactness of Y to choose a finite subset $I_{m,n}(x)$ from the indexing set $I_{m,n}$ so that $\mathcal{B}_{m,n}(x) = \{B^i_{m,n} : i \in I_{m,n}(x)\}$ is a finite cover of Y with $x \in A^i_{m,n}$ for each $i \in I_{m,n}(x)$.

Since Y is a P^*-space, for each $n \in \mathbb{N}$ we can choose an open refinement $\mathcal{D}_{m,n}(x)$ of $\mathcal{B}_{m,n}(x)$ having property P so that $\bigcup_{n \in \mathbb{N}} \mathcal{D}_{m,n}(x)$ is a cover of Y.

We use again the compactness of Y, this time to choose a positive integer $r_m(x) \in \mathbb{N}$ so that $\bigcup_{n=1}^{r_m(x)} \mathcal{D}_{m,n}(x)$ is a finite subcover of Y, and then we set $A_m(x) = \bigcap_{n=1}^{r_m(x)} \{A^i_{m,n} : i \in I_{m,n}(x)\}$. Since $I_{m,n}(x)$ is a finite set, $A_m(x)$ is an open neighborhood of x in X. We form the open cover $\mathcal{A}_m = \{A_m(x) : x \in X\}$ of X by constructing such a neighborhood $A_m(x)$ for each $x \in X$.

In this manner, we construct such an open cover $\mathcal{A}_m$ of X for each $m \in \mathbb{N}$. Since X is a P^*-space, we can choose a refinement $\mathcal{C}_m$ of $\mathcal{A}_m$ for each $m \in \mathbb{N}$ having the property P so that $\bigcup_{m \in \mathbb{N}} \mathcal{C}_m$ covers X. Since each $\mathcal{C}_m$ refines $\mathcal{A}_m$, we can choose a function $\phi_m : \mathcal{C}_m \to X$ for each $m \in \mathbb{N}$ so that if $C \in \mathcal{C}_m$ we have $C \subset A_m(\phi_m(C))$.

Thus, if $n \in \{1, \ldots, r_m(\phi_m(C))\}$ for some $C \in \mathcal{C}_m$, then $C \subset A_m(\phi_m(C))$, and thus for any $n \in \{1, \ldots, r_m(\phi_m(C))\}$ and $i \in I_{m,n}(x)$ we have $C \subset A_{m,n}^i$.

Now, for each fixed $m, n \in \mathbb{N}$, we define

$$\mathcal{V}_{m,n} = \{C \times D : C \in \mathcal{C}_m, D \in \mathcal{D}_{m,n}(\phi_m(C))\}.$$

If $C \times D \in \mathcal{V}_{m,n}$, then $C \in \mathcal{C}_m$ with $D \in \mathcal{D}_{m,n}(\phi_m(C))$, and thus $D \subset B_{m,n}^i$ for some $i \in I_{m,n}(\phi_m(C))$. Therefore, we see that $C \times D \subset A_{m,n}^i \times B_{m,n}^i \in \mathcal{U}_{m,n}$, so that $\mathcal{V}_{m,n}$ is an open refinement of $\mathcal{U}_{m,n}$.

Moreover, since every $\mathcal{C}_m$ and every $\mathcal{D}_{m,n}(x)$ have property P, then $\mathcal{V}_{m,n}$ has P, for each $m, n \in \mathbb{N}$, from the productivity of P.

Finally, since $\bigcup_{m \in \mathbb{N}} \mathcal{C}_m$ covers X, given a point $(x, y) \in X \times Y$ we can find $m \in \mathbb{N}$ and $C \in \mathcal{C}_m$ so that $x \in C$. Since Y is covered by $\bigcup_{n=1}^{r_m(\phi_m(C))} \mathcal{D}_{m,n}(\phi_m(C))$, we can also find $n \in \{1, \ldots, r_m(\phi_m(C))\}$ and $D \in \mathcal{D}_{m,n}(\phi_m(C))$ so that $y \in D$. Hence $(x, y) \in C \times D$, so that $\bigcup_{m,n \in \mathbb{N}} \mathcal{V}_{m,n}$ covers $X \times Y$, so we conclude that $X \times Y$ is P^*. $\square$

Note that there are properties P where compact implies P^* (as finiteness and local finiteness) so one can replace P^*-compact with compact, obtaining for example that the product of a compact space with a paracompact (Hurewicz) space is again paracompact (Hurewicz).

On the other hand, it is interesting to remark the following: suppose one have three topological properties satisfied by a family of subsets of a topological space in such a way that whenever $\mathcal{U}$ is a family of subsets of a set X having property P and for each $\mathcal{U}$ there is a family $\mathcal{V}(\mathcal{U})$ of subsets of a set Y having property Q, the family $\{U \times V : U \in \mathcal{U}$ and $V \in \mathcal{V}(\mathcal{U})\}$ has property R. Then the same proof of the above theorem gives the following corollary.

Corollary 2.2. *Let P, Q and R be topological properties as above. The product of a P^*-space with a Q^*-compact is a R^*-space.*

We consider now when P^* inherits to subspaces. Since compactness is such a property and compactness only inherits to closed sets, we only can expect that P^* inherits to closed sets, as our last result shows.

Definition 3. Let P be a topological property satisfied by a family of subsets of a topological space. We say that P is hereditary if whenever $\mathcal{U}$ is a family of subsets of a set X having property P, the family $\mathcal{U} \cap C = \{U \cap C : U \in \mathcal{U}\}$ has property P, for every closed C.

Theorem 2.3. *Let P be a hereditary property of a family of subsets of a topological space. If X is P^*, any closed subset of X is also P^*.*

PROOF. Any sequence of open coverings of a closed subset C is of the form $\mathcal{U}_n = \{U_i^n : i \in I_n\}$, $n \in \mathbb{N}$, where $U_i^n = W_i^n \cap C$ and W_i^n is open in X, for every $n \in \mathbb{N}$, $i \in I_n$.

Take the sequence of coverings of X $\mathcal{W}_n = \{W_i^n : i \in I_n\} \cup \{X \setminus C\}$, apply that X is P^* to obtain a sequence of open families of X having property P such that its union is a covering of X. Intersecting those families with the closed set C and applying that the property is hereditary, the result follows inmediately. $\qquad\square$

References

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