On n-distributive system of elements of a modular lattice

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1. Introduction

A modular lattice is called n-distributive [2] if it satisfies the identity

$$D_n\colon\ x\wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n}\Big[x\wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i\Big].$$

In [2] we proved the following

Theorem A. *Let M be a modular lattice. Then the following conditions are equivalent to each other and to their duals:*
(i) *M satisfies D_n.*
(ii) *M satisfies the identity*

$$M_n\colon\ \bigwedge_{j=0}^{n+1}\ \bigvee_{\substack{i=0\\ i\neq j}}^{n+1} y_i = \bigvee_{k=0}^{n+1}\ \bigwedge_{\substack{j=0\\ j\neq k}}^{n+1}\ \bigvee_{\substack{i=0\\ i\neq j,k}}^{n+1} y_i.$$

(For the "n-tributive" identity M_n see also BERGMAN [1].)

It is well-known that in the classical case $n=1$ there is a stronger, local version of the theorem:

Theorem B. *For any modular lattice M and any elements $x, y, z\in M$, the following conditions are equivalent to each other and to their duals:*

(i) $$x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z).$$

(ii) $$x\pi\wedge(y\pi\vee z\pi) = (x\pi\wedge y\pi)\vee(x\pi\wedge z\pi)$$

for any permutation π of the set $\{x, y, z\}$.

(iii) $$(x\wedge y)\vee(x\wedge z)\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)\wedge(y\vee z).$$

The purpose of this paper is to generalize Theorem B to a local version of Theorem A.

Definition 1. An ordered system $(y_0, y_1, \dots, y_{n+1})$ of elements of a modular lattice is called to be an n-distributive system if the following relation holds:

$$y_{n+1}\wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n}\Big[y_{n+1}\wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i\Big].$$

This situation will be denoted by $D_n(y_0, y_1, \dots, y_{n+1})$. The system $(y_0, y_1, \dots, y_{n+1})$ is called n-tributive if the relation

$$M_n(y_0, y_1, \dots, y_{n+1}) \colon \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j}}^{n+1} y_i = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j\neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j,k}}^{n+1} y_i$$

holds. The relations dual to D_n (resp. M_n) will be denoted by D_n^* (resp. M_n^*).

Now we can formulate the two main results of this paper.

Theorem 1. *For any modular lattice M and for arbitrary elements*

$$y_0, y_1, \dots, y_{n+1} \in M$$

the following conditions are equivalent:

(A) $$D_n(y_0, y_1, \dots, y_{n+1}).$$

(B) $$D_n(y_0\pi, y_1\pi, \dots, y_{n+1}\pi)$$

for any permutation π of the set $\{y_0, y_1, \dots, y_{n+1}\}$.

(C) $$M_n(y_0, y_1, \dots, y_{n+1}).$$

Theorem 2. *Let M be a modular lattice and let $y_0, y_1, \dots, y_{n+1} \in M$. Set*

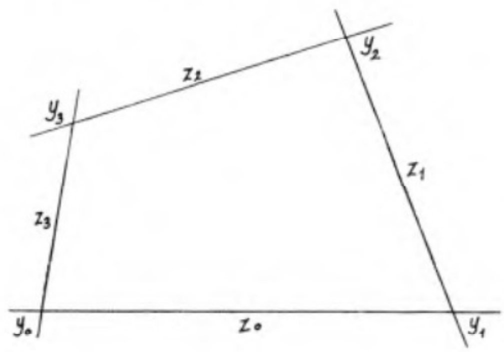
$$z_j = \bigvee_{\substack{i=0 \\ i\not\equiv j-1,\, j-2 \pmod{n+2}}}^{n+1} y_i \quad (j = 0, 1, \dots, n+1).$$

Then the following conditions are equivalent:

(A) $$D_n(y_0, y_1, \dots, y_{n+1}).$$

(D) $$D_n^*(z_0, z_1, \dots, z_{n+1}).$$

Remark. The geometrical background of this theorem is the following obvious statement: If the four vertices of a square in a projective plane constitute a system of general position then the four sides of the square are also of general position and conversely (see Figure). Indeed, if the lattice M of Theorem 2 is the subspace lattice of a projective plane, then the above statement and Theorem 2 coincide.



To prove this theorems we need to generalize the technique used in [2]. First of all we generalize the notion of n-diamond (cf. [3]).

Definition 2. An ordered system $(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n+1})$ of elements of a modular lattice is called a $\wedge$-n-diamond if for any choice of the $(n+1)$-element subsets $\{j_0, j_1, \ldots, j_n\}$ and $\{k_0, k_1, \ldots, k_n\}$ of the index set $\{0, 1, \ldots, n+1\}$ the relation

$$\bar{b}_{j_0} \wedge \bigvee_{i=1}^{n} \bar{b}_{j_i} = \bar{b}_{k_0} \wedge \bigvee_{i=1}^{n} \bar{b}_{k_i}$$

holds. $(\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_{n+1})$ is called a $\vee$-n-diamond if for any choice of the $(n+1)$-element subsets $\{j_0, j_1. \ldots, j_n\}$ and $\{k_0, k_1, \ldots, k_n\}$ of the index set $\{0, 1, \ldots, n+1\}$

$$\bigvee_{i=0}^{n} \underline{b}_{j_i} = \bigvee_{i=0}^{n} \underline{b}_{k_i}$$

holds. If $([\underline{b}_0, \bar{b}_0], [\underline{b}_1, \bar{b}_1], \ldots, [\underline{b}_{n+1}, \bar{b}_{n+1}])$ is an ordered system of intervals of a modular lattice so that $(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n+1})$ is a $\wedge$-n-diamond and $(\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_{n+1})$ is a $\vee$-n-diamond, then we call this system a generalized n-diamond. A generalized n-diamond (resp., $\wedge$-n-diamond, $\vee$-n-diamond) is called non-trivial if its elements are pairwise disjoint (resp., distinct).

The basic tool in proving Theorems 1 and 2 will be the following

Theorem 3. *For any modular lattice M and elements $y_0, y_1, \ldots, y_{n+1} \in M$ the following two conditions are equivalent:*

(I) *$D_n(y_0, y_1, \ldots, y_{n+1})$ does not hold.*

(II) *There exists a non-trivial generalized n-diamond $([\underline{b}_0, \bar{b}_0], \ldots, [\underline{b}_{n+1}, \bar{b}_{n+1}])$ in M such that $y_i \in [\underline{b}_i, \bar{b}_i]$ for $i = 0, 1, \ldots, n+1$.*

2. Proof of Theorem 3

First we recall the following statement from [2].

Lemma 1. *Let M be a modular lattice and let $y_0, y_1, \ldots, y_{n+1} \in M$ such that*

$$y_{n+1} \wedge \bigvee_{i=0}^{n} y_i \neq \bigvee_{j=0}^{n} \Big[y_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i\Big].$$

Set

$$w_0 = y_{n+1} \wedge \bigvee_{i=0}^{n} y_i, \quad u_0 = \bigvee_{j=0}^{n} \Big[y_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i\Big]$$

$$y'_j = \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i \quad (j = 0, 1, \ldots, n)$$

$$v = \bigwedge_{j=0}^{n} (w_0 \vee y'_j), \quad b'_j = (y'_j \vee u_0) \wedge v \quad (j = 0, 1, \ldots, n),$$

$$b_i = \bigwedge_{\substack{j=0 \\ j \neq i}}^{n} b'_j \quad (i = 0, 1, \ldots, n), \quad u = \bigwedge_{j=0}^{n} b'_j, \quad w = u \vee w_0.$$

Then the elements b_i generate a Boolean sublattice B of length $n+1$ in B such that u is the g.l.b., v is the l.u.b. of B and w is a common relative complement of the dual atoms of B in $[u, v]$.

Lemma 2. *The elements u and v defined in Lemma 1 can be written in the form*

$$u = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0\\ j\neq k}}^{n+1} \bigvee_{\substack{i=0\\ i\neq j,k}}^{n+1} y_i, \quad v = \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0\\ i\neq j}}^{n+1} y_i.$$

PROOF. Compute:

$$v = \bigwedge_{j=0}^{n} (w_0 \vee y_j') = \bigwedge_{j=0}^{n} \Big[(y_{n+1} \wedge \bigvee_{i=0}^{n} y_i) \vee \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i\Big] =$$

$$= \bigwedge_{j=0}^{n} \Big[(y_{n+1} \vee \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i) \wedge \bigvee_{i=0}^{n} y_i\Big] = \bigwedge_{j=0}^{n} \bigvee_{\substack{i=0\\ i\neq j}}^{n+1} y_i \wedge \bigvee_{\substack{i=0\\ i\neq n+1}}^{n+1} y_i = \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0\\ i\neq j}}^{n+1} y_i.$$

$$u = \bigwedge_{j=0}^{n} b_j' = \bigwedge_{j=0}^{n} [(y_j' \vee u_0) \wedge v] = v \wedge \bigwedge_{j=0}^{n} (y_j' \vee u_0).$$

Now we write $y_j' \vee u_0$ in a more suitable form.

$$y_j' \vee u_0 = u_0 \vee y_j' = \bigvee_{k=0}^{n} \Big[y_{n+1} \wedge \bigvee_{\substack{l=0\\ l\neq k}}^{n} y_l\Big] \vee \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i =$$

$$= \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big[y_{n+1} \wedge \bigvee_{\substack{l=0\\ l\neq k}}^{n} y_l\Big] \vee \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i = \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big[(y_{n+1} \wedge \bigvee_{\substack{l=0\\ l\neq k}}^{n} y_l) \vee \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i\Big] =$$

$$= \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big[(y_{n+1} \wedge \bigvee_{\substack{l=0\\ l\neq k}}^{n} y_l) \vee \bigvee_{\substack{i=0\\ i\neq j,k}}^{n} y_i \vee y_k\Big] = \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big\{\Big[(y_{n+1} \vee \bigvee_{\substack{i=0\\ i\neq j,k}}^{u} y_i) \wedge \bigvee_{\substack{l=0\\ l\neq k}}^{n} y_l\Big] \vee y_k\Big\} =$$

$$= \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big\{\Big[(y_{n+1} \vee \bigvee_{\substack{i=0\\ i\neq j,k}}^{n} y_i) \wedge (y_j \vee \bigvee_{\substack{l=0\\ l\neq j,k}}^{n} y_l)\Big] \vee y_k\Big\} =$$

$$= \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big\{\Big[(y_{n+1} \vee \bigvee_{\substack{i=0\\ i\neq j,k}}^{n} y_i) \wedge y_j\Big] \vee \Big[\bigvee_{\substack{l=0\\ l\neq j,k}}^{n} y_l \vee y_k\Big]\Big\} =$$

$$= \bigvee_{\substack{l=0\\ l\neq j}}^{n} y_l \vee \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big[(y_{n+1} \vee \bigvee_{\substack{i=0\\ i\neq j,k}}^{n} y_i) \wedge y_j\Big].$$

Then

$$u = v \wedge \bigwedge_{j=0}^{n} \Big\{\bigvee_{\substack{l=0\\ l\neq j}}^{n} y_l \vee \bigvee_{\substack{k=0\\ k\neq j}}^{n} \Big[(y_{n+1} \vee \bigvee_{\substack{i=0\\ i\neq j,k}}^{n} y_i) \wedge y_j\Big]\Big\}.$$

In the above expression $\bigvee_{k=0, k\neq j}^{n} [(y_{n+1} \vee \bigvee_{i=0, i\neq j,k}^{n} y_i) \wedge y_j] \leqq \bigvee_{l=0, l\neq j}^{n}, y_l$ if $j \neq j'$ thus we can use the following form of the modular identity:

$$\bigvee_{i=1}^{r} p_i \vee \bigwedge_{i=1}^{r} q_i = \bigwedge_{i=1}^{r} (p_i \vee q_i) \tag{1}$$

provided $p_i \leqq q_j$ if $i \neq j$. Hence we obtain

$$u = v \wedge \{ \bigwedge_{j=0}^{n} \bigvee_{\substack{l=0 \\ l\neq j}}^{n} y_l \vee \bigvee_{j=0}^{n} \bigvee_{\substack{k=0 \\ k\neq j}}^{n} [(y_{n+1} \vee \bigvee_{\substack{i=0 \\ i\neq j,k}}^{n} y_i) \wedge y_j] \}.$$

Now write the join $\bigvee_{j=0}^{n} \bigvee_{k=0, k=j}^{n} [\dots]$ in the form $\bigvee_{k=0}^{n} \bigvee_{j=0, j\neq k}^{n} [\dots]$ and apply the dual of (1) to the join $\bigvee_{j=0, j\neq k}^{n} [\dots]$. Then

$$u = v \wedge \{ \bigwedge_{j=0}^{n} \bigvee_{\substack{l=0 \\ l\neq j}}^{n} y_l \vee \bigvee_{k=0}^{n} [\bigvee_{\substack{j=0 \\ j\neq k}}^{n} y_j \wedge \bigwedge_{\substack{j=0 \\ j\neq k}}^{n} (y_{n+1} \vee \bigvee_{\substack{i=0 \\ i\neq j,k}}^{n} y_i)] \} =$$

$$= v \wedge \{ \bigwedge_{j=0}^{n} \bigvee_{\substack{l=0 \\ l\neq j}}^{n} y_l \vee \bigvee_{k=0}^{n} \bigwedge_{\substack{j=0 \\ j\neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j,k}}^{n+1} y_i \} = v \wedge \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j\neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j,k}}^{n+1} y_1 =$$

$$= \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j}}^{n+1} y_i \wedge \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j\neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j,k}}^{n+1} y_i = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j\neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j,k}}^{n+1} y_i,$$

as claimed.

Lemma 3. *Let M be a modular lattice, $y_0, y_1, \dots, y_{n+1} \in M$, and let $\bar{b}_i = y_i \vee u$. $\underline{b}_i = y_i \wedge v$ $(i=0, 1, \dots, n+1)$. Then $(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1})$ is a $\wedge$-n-diamond and g.l.b, $\{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\} = u$, $(\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1})$ is a $\vee$-n-diamond and l.u.b. $\{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\} = v$.*

Proof. It suffices to show that

$$(y_0 \vee u) \wedge (\bigvee_{i=1}^{n} (y_i \vee u)) = u \tag{2}$$

and

$$\bigvee_{i=0}^{n} (y_i \wedge v) = v. \tag{3}$$

Proof of (2). First we compute $\bigwedge_{j, j\neq k} \bigvee_{i, i\neq j,k} y_i$ for all $k (=0, 1, \dots, n+1)$.

(a) If $k=0$, then by the dual of (1)

$$\bigwedge_{\substack{j=0 \\ j\neq 0}}^{n+1} \bigvee_{\substack{i=0 \\ i\neq j,0}}^{n+1} y_i = \bigvee_{\substack{i=0 \\ i\neq 0,n+1}}^{n+1} y_i \wedge \bigwedge_{\substack{j=0 \\ j\neq 0,n+1}}^{n+1} \bigvee_{\substack{i=0 \\ i\neq 0,j}}^{n+1} y_i = \bigvee_{\substack{j=0 \\ j\neq 0,n+1}}^{n+1} [y_j \wedge \bigvee_{\substack{i=0 \\ i\neq 0,j}}^{n+1} y_i].$$

(b) If $k=n+1$, then similarly

$$\bigwedge_{\substack{j=0\\ j\neq n+1}}^{n+1} \bigvee_{\substack{i=0\\ i\neq j,n+1}}^{n+1} y_i = \bigvee_{\substack{j=0\\ j\neq 0,n+1}}^{n+1} \Big[y_j \wedge \bigvee_{\substack{i=0\\ i\neq j,n+1}}^{n+1} y_i\Big].$$

(c) If $k\neq 0, n+1$, then

$$\bigwedge_{\substack{j=0\\ j\neq k}}^{n+1} \bigvee_{\substack{i=0\\ i\neq j,k}}^{n+1} y_i = \bigvee_{\substack{j=0\\ j\neq k,n+1}}^{n+1} \Big[y_j \wedge \bigvee_{\substack{i=0\\ i\neq j,k}}^{n+1} y_i\Big] =$$

$$= \bigvee_{\substack{j=0\\ j\neq 0,k,n+1}}^{n+1} \Big[y_j \wedge \bigvee_{\substack{i=0\\ i\neq j,k}}^{n+1} y_i\Big] \vee \Big[y_0 \wedge \bigvee_{\substack{i=0\\ i\neq 0,k}}^{n+1} y_i\Big].$$

(d) Let $y_j^{(k)} = y_j \wedge \bigvee_{\substack{i=0\\ i\neq j,k}}^{n+1} y_i$. Then, by Lemma 2,

$$u = \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0\\ j\neq k}}^{n+1} \bigvee_{\substack{i=0\\ i\neq j,k}}^{n+1} y_i =$$

$$= \Big[\bigvee_{\substack{j=0\\ j\neq 0\\ j\neq n+1}}^{n+1} y_j^{(0)} \vee \bigvee_{\substack{j=0\\ j\neq 0\\ j\neq n+1}}^{n+1} y_j^{(n+1)} \vee \bigvee_{\substack{k=0\\ k\neq 0\\ k\neq n+1}}^{n+1} \bigvee_{\substack{j=0\\ j\neq 0\\ j\neq n+1\\ j\neq k}}^{n+1} y_j^{(k)}\Big] \vee \bigvee_{\substack{k=0\\ k\neq 0\\ k\neq n+1}}^{n+1} y_0^{(k)}.$$

Now let this latter join be abbreviated by $A\vee B$. Then it is easy to see that $A \leqq \bigvee_{j,\, j\neq 0,\, j\neq n+1} y_j$ and $B \leqq y_0$. Thus again by the dual of (1)

$$(y_0 \vee A \vee B) \wedge \Big(\bigvee_{\substack{j=0\\ j\neq 0,n+1}}^{n+1} y_j \vee A \vee B\Big) = (y_0 \vee A) \wedge \Big(\bigvee_{\substack{j=0\\ j\neq 0,n+1}}^{n+1} y_j \vee B\Big) =$$

$$= \Big(y_0 \wedge \bigvee_{\substack{j=0\\ j\neq 0,n+1}}^{n+1} y_j\Big) \vee A \vee B = y_0^{(n+1)} \vee A \vee B = y_0^{(n+1)} \vee u = u,$$

since for all $y_j^{(k)}, y_j^{(k)} \leqq u$.

PROOF OF (3).

$$\bigvee_{\substack{i=0\\ i\neq n+1}}^{n+1} (y_i \wedge v) = \bigvee_{\substack{i=0\\ i\neq n+1}}^{n+1} \Big(y_i \wedge \bigwedge_{j=0}^{n+1} \bigvee_{\substack{l=0\\ l\neq j}}^{n+1} y_l\Big) = \bigvee_{\substack{i=0\\ i\neq n+1}}^{n+1} \Big(y_i \wedge \bigvee_{\substack{l=0\\ l\neq i}}^{n+1} y_l\Big) =$$

$$= \bigvee_{\substack{i=0\\ i\neq n+1}}^{n+1} y_i \wedge \bigwedge_{\substack{i=0\\ i\neq n+1}}^{n+1} \bigvee_{\substack{l=0\\ l\neq i}}^{n+1} y_i = \bigwedge_{i=0}^{n+1} \bigvee_{\substack{l=0\\ l\neq i}}^{n+1} y_i = v.$$

This completes the proof of the Lemma.

PROOF OF THEOREM 3, (I)$\Rightarrow$(II). The intervals $[\underline{b}_0, \bar{b}_0], [\underline{b}_1, \bar{b}_1], \dots, [\underline{b}_{n+1}, \bar{b}_{n+1}]$, formed from the elements $\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}$ and $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}$ defined in Lemma 3, satisfy the conditions given in (II), provided they are disjoint. Assume that there exist integers $l, m \in \{0, 1, \dots, n+1\}$ such that $[\underline{b}_l, \bar{b}_l] \cap [\underline{b}_m, \bar{b}_m] \neq \emptyset$. Then $\underline{b}_l \leqq \leqq \bar{b}_l \wedge \bar{b}_m = u$. Hence, computing by Lemma 3,

$$v = \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} \underline{b}_i = \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \Big(\bigvee_{\substack{i=0 \\ i \neq j,l}}^{n+1} \underline{b}_i \vee \underline{b}_l \Big) \wedge \bigvee_{\substack{i=0 \\ i \neq l}}^{n+1} \underline{b}_i \leqq$$

$$\leqq \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \Big(\bigvee_{\substack{i=0 \\ i \neq j,l}}^{n+1} \underline{b}_i \vee \underline{b}_l \Big) \leqq \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \Big(\bigvee_{\substack{i=0 \\ i \neq j,l}}^{n+1} \bar{b}_i \vee u \Big) = \bigwedge_{\substack{j=0 \\ j \neq l}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j,l}}^{n+1} \bar{b}_i = u.$$

This yields $v = u$ which is a contradiction since, by Lemma 1, the length of the interval $[u, v]$ is at least $n+1$. Q.e.d.

PROOF OF THEOREM 3, (II)$\Rightarrow$(I). Let $\bar{u} = \text{g.l.b. } \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\}$ and let $\underline{v} = \text{l.u.b. } \{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\}$. Assume that both (II) and $\neg$(I) hold. Then

$$\underline{b}_{n+1} = b_{n+1} \wedge \underline{v} = \underline{b}_{n+1} \wedge \bigvee_{i=0}^{n} \underline{b}_i \leqq y_{n+1} \wedge \bigvee_{i=0}^{n} y_i =$$

$$= \bigvee_{j=0}^{n} \Big[y_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i \Big] \leqq \bigvee_{j=0}^{n} \Big[\bar{b}_{n+1} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} \bar{b}_i \Big] = \bar{u}.$$

Hence, by a similar computation as in the proof of (I)$\Rightarrow$(II) we obtain $\underline{v} \leqq \bar{u}$. This yields that $\underline{b}_i \leqq \underline{v} \leqq \bar{u} \leqq \bar{b}_i$ which contradicts the assumption that the intervals $[\underline{b}_i, \bar{b}_i]$ are disjoint. Q.e.d.

3. Proof of Theorems 1 and 2

PROOF OF THEOREM 1. (A)$\Leftrightarrow$(B). is obvious from Theorem 3.

(B)$\Rightarrow$(C) has been proved in [2] (see the proof of [2], Corollary 2.3).

$\neg$(A)$\Rightarrow \neg$(C). Assume that $D_n(y_0, y_1, \dots, y_{n+1})$ does not hold. Then, by Theorem 3, there is a non-trivial generalized n-diamond $([\underline{b}_0, \bar{b}_0], [\underline{b}_1, \bar{b}_1], \dots, [\underline{b}_{n+1}, \bar{b}_{n+1}])$ in M such that $\underline{b}_i \leqq y_i \leqq \bar{b}_i$ $(i = 0, 1, \dots, n+1)$. Now assume that $M_n(y_0, y_1, \dots, y_{n+1})$ holds. Then we obtain that

$$l.u.b. \ \{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\} = \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} \underline{b}_i \leqq \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i =$$

$$= \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j,k}}^{n+1} y_i \leqq \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j,k}}^{n+1} \bar{b}_i = g.l.b. \ \{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\},$$

a similar contradiction as in the proof of Theorem 3, (II)$\Rightarrow$(I).

PROOF OF THEOREM 2. Define y_i and z_j as in the Introduction and set

$$\bar{y}_i = \bigwedge_{\substack{j=0 \\ j \not\equiv i+1,\, i+2 \,(\mathrm{mod}\, n+2)}}^{n} z_j \quad (i = 0, 1, \ldots, n+1).$$

First we prove the following statement:
(a) *$D_n(y_0, y_1, \ldots, y_{n+1})$ implies $D_n(\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n+1})$.*

Indeed, $\bar{y}_i \leqq z_j$ if $j \not\equiv i+1, i+2 \,(\mathrm{mod}\, n+2)$, thus $\bar{y}_i \leqq z_j$ if $i \not\equiv j-1, j-2 \,(\mathrm{mod}\, n+2)$. Hence

$$\bigvee_{\substack{i=0 \\ i \not\equiv j-1,\, j-2 \,(\mathrm{mod}\, n+2)}}^{n+1} \bar{y}_i \leqq z_j = \bigvee_{\substack{i=0 \\ i \not\equiv j-1,\, j-2 \,(\mathrm{mod}\, n+2)}}^{n+1} y_i \quad (j = 0, 1, \ldots, n+1),$$

thus

$$\bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} \bar{y}_i \leqq \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i$$

holds for any $j(=0, 1, \ldots, n+1)$. Also $y_i \leqq \bar{y}_i$, whence

$$\bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j,k}}^{n+1} y_i \leqq \bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j,k}}^{n+1} \bar{y}_i \leqq \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} \bar{y}_i \leqq \bigwedge_{j=0}^{n+1} \bigvee_{\substack{i=0 \\ i \neq j}}^{n+1} y_i .$$

This proves (a).

Therefore, it suffices to show the following two statements:

(b) $\quad \ulcorner D_n(y_0, y_1, \ldots, y_{n+1})$ *implies* $\ulcorner D_n^*(z_0, z_1, \ldots, z_{n+1})$.

(c) $\quad \ulcorner D_n^*(z_0, z_1, \ldots, z_{n-1})$ *implies* $\ulcorner D_n(\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n+1})$.

But, by symmetry and duality, (c) follows from (b), thus it is sufficient to prove only (b).

Now assume that $D_n(y_0, y_1, \ldots, y_{n+1})$ does not hold. Then there is a nontrivial generalized n-diamond $([\underline{b}_0, \bar{b}_0], [\underline{b}_1, \bar{b}_1], \ldots, [\underline{b}_{n+1}, \bar{b}_{n+1}])$ in M such that $\underline{b}_i \leqq y_i \leqq \bar{b}_i$. We finish the proof in the following three steps.

1. Obviously $z_j \leqq \bigvee_{i=0,\, i \not\equiv j-1,\, j-2 \,(\mathrm{mod}\, n+?)}^{n+1} \bar{b}_i$. Hence, computing in the generalized n-diamond, we obtain that $\bigwedge_{j=0,\, j \neq k}^{n+1} z_j \leqq$ g.l.b. $\{\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n+1}\}$. Thus

$$\bigvee_{k=0}^{n+1} \bigwedge_{\substack{j=0 \\ j \neq k}}^{n+1} z_j \leqq g.l.b.\ \{\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n+1}\}.$$

2. We prove that $\bigwedge_{l=0}^{n+1} \bigvee_{k=0,\, k \neq l}^{n+1} \bigwedge_{j=0,\, j \neq k,l}^{n+1} z_j \geqq$ l.u.b. $\{\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_{n+1}\}$. It is easy to show (see, for instance, the proof of [2] Corollary 2.3), that

$$\bigwedge_{k=0}^{n+1} \bigvee_{\substack{j=0 \\ j \neq k}}^{n+1} \bigwedge_{\substack{i=0 \\ i \neq j,k}}^{n+1} z_i = \bigwedge_{j=0}^{n+1} \bigwedge_{\substack{k=0 \\ k \neq j}}^{n+1} \Big[z_j \vee \bigwedge_{\substack{i=0 \\ i \neq j,k}}^{n+1} z_i .\Big]$$

Thus, by symmetry, it suffices to show that

$$z_0 \vee \bigwedge_{\substack{j=1 \\ j \neq l}}^{n+1} z_j \geqq l.u.b.\ \{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\}$$

holds for all $l(=1, 2, \dots, n+1)$. This is trivial if $l=1$ or $l=n+1$. Indeed, if, for instance, $l=1$, then

$$z_0 \vee \bigwedge_{j=2}^{n+1} z_j \geqq \bigvee_{i=0}^{n-1} y_i \vee y_{n+1} \geqq l.u.b.\ \{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\}.$$

Assume that $l \neq 1, n+1$. Then

$$z_0 \vee \bigwedge_{\substack{j=1 \\ j \neq l}}^{n+1} z_j = z_0 \vee [\bigwedge_{j=1}^{l-1} z_j \wedge \bigwedge_{j=l+1}^{n+1} z_j] \geqq$$

$$\geqq \bigvee_{i=0}^{n-1} y_i \vee [\bigvee_{i=l-1}^{n} y_i \wedge (\bigvee_{i=0}^{l-2} y_i \vee y_{n+1})] =$$

$$= \bigvee_{i=0}^{l-2} y_i \vee \bigvee_{i=l-1}^{n-1} y_i \vee [\bigvee_{i=l-1}^{n} y_i \wedge (\bigvee_{i=0}^{l-2} y_i \vee y_{n+1})] =$$

$$= [\bigvee_{i=0}^{l-2} y_i \vee \bigvee_{i=l-1}^{n} y_i] \wedge [\bigvee_{i=l-1}^{n-1} y_i \vee \bigvee_{i=0}^{l-2} y_i \vee y_{n+1}] \geqq$$

$$\geqq l.u.b.\ \{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\},$$

as claimed.

3. Because of the two previous observations $D_n^*(z_0, z_1, \dots, z_{n+1})$ would mean that g.l.b. $\{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\}=$l.u.b. $\{\underline{b}_0, \underline{b}_1, \dots, \underline{b}_{n+1}\}$, i.e., g.l.b. $\{\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{n+1}\} \in [\underline{b}_i, \bar{b}_i]$ for all i $(i=0, 1, \dots, n+1)$. This contradicts the fact that the intervals $[\underline{b}_i, \bar{b}_i]$ are disjoint. Thus $D_n^*(z_0, z_1, \dots, z_{n+1})$ does not hold which completes the proof.

References

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(Received March 28, 1977)