Abelian groups, nil modulo a subgroup, need not have nil quotient group

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1. Introduction

For the purposes of this paper all groups are abelian and written additively. A ring R is said to be a ring on G if the group G is isomorphic to the additive group of R. In this situation we write $R=(G, *)$ where $*$ denotes the ring multiplication. This multiplication is not assumed to be associative.

Every group may be turned into a ring in a trivial way by setting all products equal to zero. If this is the only multiplication compatible with G then G is said to be a *nil group*. For example every divisible torsion group is a nil group and these are the only torsion nil groups (see [3] or [2] for details).

A generalisation of the notion of nil group is considered by FEIGELSTOCK [1]. Suppose that H is a subgroup of G; *G is nil modulo H* if $G*G\subseteqq H$ for every ring $(G, *)$ on G. Clearly G is a nil group if and only if G is nil modulo 0. FEIGELSTOCK [1] shows that if H is a divisible subgroup of G and G is nil modulo H, then G/H is a nil group. Feigelstock goes on to ask if this is true in general. In other words:
(1.1) *Does G nil modulo H imply that G/H is a nil group?*

In this note we show that the answer to this question must be no. However, the question has a positive answer if G is a torsion group, or if H is a direct summand of G. We go on to pose a refined version of the question which is strongly related to the problem of characterising those subgroups of G which must be ideals in any ring defined on G. We note here that any fully invariant subgroup of G is an ideal in any ring $(G, *)$ [2, p. 279.]

Our results are independant of the associative nature of the rings considered. In order to cope with the difficulty that the hypothesis that G be a nil group seems much stronger than the hypothesis that G has no nontrivial associative rings defined on it we introduce two further definitions.

(i) G is an *Anil* group if the only associative ring $(G, *)$ on G is the trivial ring, $G*G=0$.

(ii) If H is a subgroup of G then *G is Anil modulo H* if given any associative ring $(G, *)$ we have $G*G\subseteqq H$.

2. Positive answers

Lemma 2.1. *Let K be a non-zero summand of G. Let $(K, \circ)$ be a nontrivial ring on K. Then $(K, \circ)$ may be imbedded as a ring direct summand in some ring $(G, *)$ in such a way that $G*G=K\circ K$. Furthermore if $(K, \circ)$ is associative then $(G, *)$ may be assumed associative.*

PROOF. Suppose that $G=H\oplus K$. We define the ring $(G, *)$ by putting,

$$(h, k)*(h', k') = (0, k\circ k')$$

The stated properties of $(G, *)$ are easily verified.

Corollary 2.2. *Let H be a direct summand of G. If G is nil (Anil) modulo H, then G/H is a nil (Anil) group.*

PROOF. Suppose that $G=H\oplus K$, where $K\cong G/H$. If K is non-nil (non-Anil) then according to Lemma 2.1 there is a nontrivial ring $(G, *)$ with $G*G\subseteqq K$, contradicting the hypothesis on G.

Corollary 2.3. (Feigelstock [1]). *If H is a divisible subgroup of G and G is nil (Anil) modulo H then G/H is nil (Anil).*

Theorem 2.4. *Let G be a torsion group. If G is nil (Anil) modulo H then G/H is nil (Anil).*

PROOF. We assume first that G is a p-group. Let B be a basic subgroup of G with independant generators $\{a_i\colon i\in I\}$. According to Theorem 120.1 [2, page 287] any multiplication $*$ on G is uniquely determined by the values a_i*a_j for all pairs i, j from I. Furthermore, a_i*a_j may be asigned any values in G subject to the condition that the order of a_i*a_j is no greater than the minimum of the orders of a_i and a_j. In particular we may define an associative multiplication on G by setting

$$a_i*a_j=\begin{cases}0 & \text{if} \quad i\neq j,\\ a_i & \text{if} \quad i=j.\end{cases} \tag{2.5}$$

It is clear that $B\subseteqq G*G$. Thus if G is nil (Anil) modulo H then $B\subseteqq H$. It follows that G/H is a quotient group of the divisible group G/B. Whence G/H is divisible and so nil.

Next we suppose that G is an arbitary reduced torsion group. Let $G=\bigoplus_p G_p$ be the decomposition of G into its primary components. The G_p are fully invariant subgroups of G and so ideals in any ring $(G, *)$. Indeed, we have a ring direct sum decomposition,

$$(G, *) = \dot{+}_p(G_p, *_p) \tag{2.6}$$

where $*_p$ is the restriction of $*$ to G_p. Conversely given any multiplications $*_p$ on the G_p we can define a ring on G via (2.6). We may also decompose H as a direct sum $\bigoplus_p H_p$ of its primary components, $H_p=H\cap G_p$. The hypothesis that G is nil (Anil) modulo H implies that each G_p is nil (Anil) modulo H_p. It follows that G_p/H_p is divisible for all primes p. Thus $G/H=\bigoplus_p (G_p/H_p)$ is divisible and so nil.

Finally we suppose that G is an arbitary torsion group. Then $G=A\oplus D$ where A is reduced and D is divisible. If $A=0$ then G and all its quotients are nil groups. We assume then that $A\neq 0$. Let $(A, \circ)$ be a ring on A, then by Lemma 2.1 there is a ring $(G, *)$ with $G*G=A\circ A$. The hypothesis that G be nil modulo H implies that,

$$A\circ A\subseteqq A\cap H.$$

Since $(A, \circ)$ is an arbitrary ring an A we see that A is nil modulo $A \cap H$. By what has gone before we deduce that $(A+H)/H \cong A/(A \cap H)$ is divisible. It follows that,

$$G/H \cong G/(A+H) \oplus (A+H)/H;$$

and since $G/(A+H)$ is clearly a quotient of D we deduce that G/H is divisible, and so a nil group.

3. A counterexample

According to [2, p. 292; Theorem 121.1 and Proposition 121.2] there is an abundance of torsion-free nil groups (in fact it is not hard to produce such a group with any preassigned cardinality). Let G be such a group. We note that G cannot be divisible and so there is an integer n for which $G \neq nG$. Thus the group G/nG is a non-trivial bounded group; and so, by a result of Szele [3] (see [2], p. 288; Theorem 120.33), G/nG is non-nil. We have then a nil-group G, which is thus nil modulo nG, for which G/nG is non-nil. Hence (1.1) must be answered in the negative. We would like to thank the referee for pointing out this simple example.

4. A reformulation of the problem

The counterexample of § 3 depends on the trivial fact that a nil group is nil modulo any of its subgroups. It is clear that if G is nil modulo both H_1 and H_2 then G is nil modulo $H_1 \cap H_2$. This suggests the following definition of the *square* subgroup K of G as,

$$K = \bigcap \{H \leqq G : G \text{ is nil modulo } H\}.$$

Clearly K is the smallest subgroup with the property that G is nil modulo K. In particular if G is a nil group, then $K=0$. We ask if G/K is a nil group?

We recall that a ring $(G, *)$ defines an element of $\mathrm{Hom}\,(G, E(G))$, $E(G)$ being the endomorphism ring of G, as follows. For each g in G, $\Theta(g)$ is the endomorphism of G defined by,

$$\Theta(g): x \to x * g \tag{4.1}$$

Conversely given $\Theta \in \mathrm{Hom}\,(G, E(G))$ we may use (4.1) to define a ring $(G, *)$. (For details see [2, p. 278 ff.])

Following Fuchs [2] we let $\mathfrak{l}(G)$ be the subgroup of $E(G)$ generated by $\mathrm{Im}\,\Theta$, for all $\Theta \in \mathrm{Hom}\,(G, E(G))$. This is an ideal of $E(G)$, with the property that a subgroup H of G is an ideal in every ring on G if and only if $\mathfrak{l}(G)H \leqq H$. (See [4], and [2] p. 279.) We remark that G is an $E(G)$ module and so $\mathfrak{l}(G)G$ is a submodule of G.

Proposition 4.1. $K = \mathfrak{l}(G)G$.

PROOF. Since G is nil modulo K, we have $\Theta(g)x = x*g \in K$ for all $\Theta \in \mathrm{Hom}\,(G, E(G))$, and g, x in G. Thus $\mathfrak{l}(G)G \subseteq K$.

Conversely, the fact that $\mathfrak{l}(G)$ is an ideal in $E(G)$ implies that G is nil modulo $\mathfrak{l}(G)G$, whence the result.

References

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