Remarks on the exchangeable random variables

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Introduction

Our main goal is to generalize the Kolmogorov's strong law of large numbers for exchangeable random variables. In addition we give a simple proof for the basic theorem of the exchangeable random variables and investigate the exponential rate of convergence in the weak law. In the sequal we deal with infinite sequences of exchangeable random variables only.

In order to formulate the main result we have to define the concept of conditional expectation somewhat more generally than it is usual (see e.g. the note of LOÈVE [2], p. 342).

Let $(\Omega, \mathscr{A}, \mathbf{P})$ be a probability space and $\mathscr{A}_1 \subset \mathscr{A}$ an arbitrary σ-algebra. To any non-negative random variable η we may define the $\mathscr{A}_1$-measurable function $f(\omega)$ by the relation

$$\int_A \eta \, dP = \int_A f(\omega) \, dP \quad (A \in \mathscr{A}, \text{ arbitrary}).$$

The Radon—Nikodym derivative $f(\omega)$ is always defined a.s., but $f(\omega)$ can take the value $+\infty$ too. The conditional expectation $M(\eta \mid \mathscr{A}_1)$ exists if this function $f(\omega)$ is finite a.s. and then $M(\eta \mid \mathscr{A}_1) = f(\omega)$.

For arbitrary random random variable η the conditional expectation exists if both $M(|\eta|_+ \mid \mathscr{A}_1)$ and $M(|\eta|_- \mid \mathscr{A}_1)$ exist and then

$$M(\eta|\mathscr{A}_1) = M(|\eta|_+|\mathscr{A}_1) - M(|\eta|_-|\mathscr{A}_1).$$

Usual properties of the conditional expectation are valid in this general case too. Let us mention some of them being slightly different from the usual ones.

1° Let ξ be measurable with respect to $\mathscr{A}_1$ and suppose that $M(\eta \mid \mathscr{A}_1)$ exists. Then $M(\xi\eta \mid \mathscr{A}_1)$ exists and

$$M(\xi\eta|\mathscr{A}_1) = \xi M(\eta|\mathscr{A}_1).$$

2° If $M(\eta)$ exists then $M(\eta) = M(M(\eta \mid \mathscr{A}_1))$. (Here the existence of the right side does not imply the existence of $M(\eta)$).

3° Let $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \mathscr{A}$. Then the existence of $M(\eta \mid \mathscr{A}_1)$ implies that of $M(\eta \mid \mathscr{A}_2)$ and in this case

$$M(M(\eta|\mathscr{A}_2)|\mathscr{A}_1) = M(\eta|\mathscr{A}_1).$$

Basic theorem of exchangeable random variables

Let the random variables $\xi_1, \xi_2, \ldots$ be exchangeable (or with other terminology, equivalent), i.e. any k of them have the same distribution. Write $\mathscr{F}_n$ for the smallest σ-algebra for which the random variables $\xi_n, \xi_{n+1}, \ldots$ are measurable and let $\mathscr{F}=\bigcap_{n=1}^{\infty} \mathscr{F}_n$ the tail σ-algebra of the sequence.

Theorem 1. *The exchangeable random variables $\xi_1, \xi_2, \ldots$ are independent under the condition $\mathscr{F}$.*

This theorem shows that every sequence of exhangeable random variables is a mixture of independent ones. Firstly Rényi and Révész [3] proved the basic theorem for exchangeable random events, then Révész [4] gave a proof for random variables.

The lemma used to the present proof, the convergence criterion of Lebesgue for conditional expectations, can be found e.g. in [1], pp. 23—24.

Lemma. *If $\eta_n \to \eta$ and $|\eta_n| \leqq \zeta$ with $M(\zeta)<\infty$, then $M(\eta_n | \mathscr{A}_1) \to M(\eta | \mathscr{A}_1)$ with probability* 1.

Proof of Theorem 1. Denote by $\mathscr{A}_k^n$ the σ-algebra generated by $\xi_k, \xi_{k+1}, \ldots, \xi_n$. For $A_0 \in \mathscr{A}_1^{n-1}$ and $A_n \in \mathscr{A}_n^n$ we intend to prove the equality

$$P(A_0 A_n | \mathscr{F}) = P(A_0 | \mathscr{F}) P(A_n | \mathscr{F}) \quad (n>1) \tag{1}$$

which implies the conditional independence.

There exists a Borel set B for which $A_n = \xi_n^{-1}(B)$. Because of the exchangeability

$$P(A_0 A_n | \mathscr{A}_N^M) = P(A_0 \{\xi_m \in B\} | \mathscr{A}_N^M)$$

provided that $n<m<N<M$. By limits $M \to \infty$, then $N \to \infty$ we obtain $P(A_0 A_n | \mathscr{F}) = P(A_0 \{\xi_m \in B\} | \mathscr{F})$.

On the other hand

$$P(A_0 \{\xi_m \in B\} | \mathscr{F}_m) = \chi_{\{\xi_m \in B\}} P(A_0 | \mathscr{F}_m)$$

where $\chi_{\{\xi_m \in B\}}$ denotes the indicator of the event $\{\xi_m \in B\}$. Let us take conditional expectation on the both sides under the condition $\mathscr{F}$, then

$$P(A_0 A_n | \mathscr{F}) = M\big(\chi_{\{\xi_m \in B\}} P(A_0 | \mathscr{F}_m) | \mathscr{F}\big).$$

According to the lemma

$$M\big(\chi_{\{\xi_m \in B\}} [P(A | \mathscr{F}_m) - P(A | \mathscr{F})]\big) \to 0 \quad (m \to \infty),$$

namely $\chi_{\{\xi_m \in B\}} |P(A | \mathscr{F}_m) - P(A | \mathscr{F})| \leqq 1$. By simple changes we obtain

$$M\big(\chi_{\{\xi_m \in B\}} P(A_0 | \mathscr{F}) | \mathscr{F}\big) = P(A_0 | \mathscr{F}) P(\xi_m \in B | \mathscr{F}) = P(A_0 | \mathscr{F}) P(A_n | \mathscr{F}).$$

Strong law of large numbers

The total analogon of the classical Kolmogorov's theorem is the following

Theorem 2. *For the exchangeable random variables* $\xi_1, \xi_2, \dots$

$$\frac{\xi_1+\xi_2+\dots+\xi_n}{n} \to \eta \tag{2}$$

a.s. iff $M(\xi_1|\mathscr{F})$ *exists and is equal to* η.

PROOF. Sufficiency. We shall say that the sequence $\{\eta_n\}$ is a reversed martingale with respect to the decreasing sequence of σ-algebras $\mathscr{F}_n$ if

(i) $$M(\eta_n|\mathscr{F}) \quad \text{exists} \qquad \left(n = 1, 2, \dots: \mathscr{F} = \bigcap_{n=1}^{\infty} \mathscr{F}_n\right),$$

(ii) $$M(\eta_n|\mathscr{F}_{n+1}) = \eta_{n+1} \quad (n = 1, 2, \dots).$$

The convergence theorem for martingales of this type is valid as well, it can be seen by going through its usual proof. There is a minor difficulty where we get for the number β_n of upcrossings of (r_1, r_2) the inequality

$$M(\beta_n|\mathscr{F}) \leqq \frac{M(|\eta_1||\mathscr{F})+|r_1|}{r_2-r_1} = \zeta_0.$$

By notation $A_k=\{\omega\colon \zeta_0<k\}\in\mathscr{F}$ we obtain

$$\int_{A_k} M(\beta_n|\mathscr{F})\,dP = \int_{A_k} \beta_n\,dP \leqq \int_{A_k} \zeta_0\,dP \leqq k,$$

hence $\beta_n \to \beta$ on A_k and here β is finite a.s. The same is true on $\sum_{k=1}^{\infty} A_k$, i.e. on Ω, therefore

$$P(\underline{\lim}\,\eta_n < r_1 < r_2 < \overline{\lim}\,\eta_n) = 0.$$

Let $\mathscr{F}_n^*$ be the smallest σ-algebra for which the random variables $S_n=\xi_1+ +\xi_2+\dots+\xi_n$ and $\xi_{n+1}, \xi_{n+2}, \dots$ are measurable. Using the exchangeability

$$M(\xi_k|\mathscr{F}_n^*) = M(\xi_l|\mathscr{F}_n^*) \quad (k < l \leqq n),$$

therefore

$$M\left(\frac{S_n}{n}\middle|\mathscr{F}_{n+1}^*\right) = \frac{1}{n}\sum_{l=1}^{n} M(\xi_l|\mathscr{F}_{n+1}^*) = M(\xi_1|\mathscr{F}_{n+1}^*) \tag{3}$$

and

$$\frac{S_{n+1}}{n+1} = \frac{1}{n+1} M(S_{n+1}|\mathscr{F}_{n+1}^*) = \frac{1}{n+1}\sum_{l=1}^{n+1} M(\xi_l|\mathscr{F}_{n+1}^*) = M(\xi_1|\mathscr{F}_{n+1}^*). \tag{4}$$

(3) and (4) prove that $\left\{\frac{S_n}{n}\right\}$ is a reversed martingal with respect to $\{\mathscr{F}_n^*\}$, consequently

$$\frac{S_n}{n} \to M(\xi_1|\mathscr{F}^*) = \eta$$

a.s. where $\mathscr{F}^* = \bigcap_{n=1}^{\infty} \mathscr{F}_n^*$. η is $\mathscr{F}$-measurable and $\mathscr{F}_{n+1} \subset \mathscr{F}_n^*$ implies $\mathscr{F} \subset \mathscr{F}^*$, so

$$\eta = M(\eta|\mathscr{F}) = M\big(M(\xi_1|\mathscr{F}^*)|\mathscr{F}\big) = M(\xi_1|\mathscr{F}).$$

Necessity. With notations $D_n = \{\omega\colon |\xi_n| > n\}$ and

$$\eta_n = \sum_{k=1}^{n} \big(\chi_{D_k} - P(D_k|\mathscr{F})\big)$$

the sequence $\{\eta_n\}$ is martingale with respect to $\{\mathscr{A}_1^n \vee \mathscr{F}\}$. If $\mathscr{A}'$ is the smallest algebra containing $\mathscr{A}_1^n$ and $\mathscr{F}$ then every $C \in \mathscr{A}'$ has the form

$$C = \sum_{k=1}^{m} A_k B_k, \quad A_k \in \mathscr{A}_1^n, \quad B_k \in \mathscr{F}$$

and $B_1, B_2, \ldots, B_m$ are disjoint sets. By using (1) we obtain

$$\int_C P(D_{n+1}|\mathscr{F})\,dP = \sum_{k=1}^{m} \int_{A_k B_k} P(D_{n+1}|\mathscr{F})\,dP =$$

$$= \sum_{k=1}^{m} M\big(\chi_{A_k} P(B_k D_{n+1}|\mathscr{F})\big) = \sum_{k=1}^{m} M\big(M(\chi_{A_k} P(B_k D_{n+1}|\mathscr{F})|\mathscr{F})\big) =$$

$$= \sum_{k=1}^{m} M\big(P(A_k|\mathscr{F}) P(B_k D_{n+1}|\mathscr{F})\big) = \sum_{k=1}^{m} M\big(P(A_k B_k D_{n+1}|\mathscr{F})\big) =$$

$$= \sum_{k=1}^{m} P(A_k B_k D_{n+1}) = P(C D_{n+1}) = \int_C \chi_{D_{n+1}}\,dP.$$

Consequently the two integral is equal for every $C \in \mathscr{A}'$ and then for every $C \in \mathscr{A}_1^n \vee \mathscr{F}$, too, which proves the martingale property. Define another sequence

$$\tilde{\eta}_n = \begin{cases} \eta_n & \text{for} \quad \omega \in C_n \\ c+1 & \text{for} \quad \omega \notin C_n \end{cases} \quad (n = 1, 2, \ldots)$$

where $C_n = \{\omega\colon \sup_{k \leqq n} \eta_k \leqq c\}$. $\{\tilde{\eta}_n\}$ is a submartingale with respect to $\{\mathscr{A}_1^n \vee \mathscr{F}\}$, namely for $A \in \mathscr{A}_1^n \vee \mathscr{F}$

$$\int_A \tilde{\eta}_{n+1}\,dP = \int_{AC_{n+1}} \eta_{n+1}\,dP + (c+1) P(A\overline{C}_{n+1}) =$$

$$= \int_{AC_n} \eta_{n+1}\,dP - \int_{A(C_n - C_{n-1})} \eta_{n+1}\,dP + (c+1) P(A\overline{C}_{n+1})$$

and since $\eta_n \leqq c$ on the set $C_n - C_{n+1}$ and therefore here $\eta_{n+1} \leqq c+1$, it follows

$$\int_A \tilde{\eta}_{n+1}\,dP \geqq \int_{AC_n} \eta_n\,dP - (c+1) P\big(A(\overline{C}_{n+1} - \overline{C}_n)\big) + (c+1) P(A\overline{C}_{n+1}) = \int_A \tilde{\eta}_n\,dP.$$

Since $|\tilde{\eta}_n|_+ \leqq c+1$, the sequence $\{\tilde{\eta}_n\}$ is convergent and thus $\{\eta_n\}$ is convergent on the set $\{\omega\colon \sup_k \eta_k \leqq c\}$ and on its union

$$\sum_{c=1}^{\infty} \{\omega\colon \sup_k \eta_k \leqq c\} = \{\omega\colon \sup_k \eta_k < \infty\}.$$

If $\sum_{k=1}^{\infty} \chi_{A_k} < \infty$ then the condition $\omega \in \{\omega\colon \sup_k \eta_k < \infty\}$ is fulfilled, therefore η_n converges to an a.s. finite limit, consequently $\sum_{k=1}^{\infty} P(D_k|\mathscr{F}) < \infty$.

Because of the exchangeability $P(D_n|\mathscr{F}) = P(|\xi_1| > n|\mathscr{F})$ which implies the existence of $M(|\xi_1||\mathscr{F})$ like the Kolmogorov's case.

Exponential rate of convergence

The existence of the moment generating function of ξ_1 in generally does not imply the exponential rate of weak convergence of the mean. We have to suppose somewhat more.

Theorem 3. *To every* $\varepsilon > 0$ *there exists a* $q < 1$ *such that*

$$\sqrt[n]{\mathbf{P}\left(\left|\frac{\xi_1+\xi_2+\ldots+\xi_n}{n} - \eta\right| > \varepsilon\right)} \to q \tag{5}$$

(η *is a fixed random variable) iff*

(i) $M(\xi_1|\mathscr{F})$ *exists,*

(ii) *there exists a function* $R_0(t)$*, for which* $R_0'(0)$ *exists and* $R_0(0)=1$*, such that*

$$R(t|\mathscr{F}) = e^{-tM(\xi_1|\mathscr{F})} M(e^{it\xi_1}|\mathscr{F}) \leqq R_0(t)$$

in a neighbourhood of the origin.

Proof. If (5) holds then by the Borel—Cantelli lemma η is, at the same time, the strong limit of the mean, therefore, according to Theorem 2, $M(\xi_1|\mathscr{F})$ exists and $\eta = M(\xi_1|\mathscr{F})$.

For the conditional distributions the Chernoff relations hold because of the Theorem 1, therefore with abbreviation

$$P_n(x|\mathscr{F}) = \mathbf{P}\left(\frac{\xi_1+\xi_2+\ldots+\xi_n}{n} - M(\xi_1|\mathscr{F}) > x|\mathscr{F}\right)$$

$$\sqrt[n]{P_n(x|\mathscr{F})} \to \varrho(x|\mathscr{F}) \tag{6}$$

and

$$P_n(x|\mathscr{F}) \leqq \varrho^n(x|\mathscr{F}) \tag{7}$$

where

$$\varrho(x|\mathscr{F}) = \inf_t e^{-tx} R(t|\mathscr{F}).$$

By (6) and (7)

$$\sqrt[n]{M(P_n(x|\mathscr{F}))} \to \operatorname*{ess\,sup}_{\omega\in\Omega} \varrho(x|\mathscr{F})$$

and $\operatorname*{ess\,sup}_{\omega\in\Omega} \varrho(x|\mathscr{F}) < 1$, for every $x \neq 0$ iff (ii) is satisfied.

References

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