Report on the
Second International Symposium on

FUNCTIONAL EQUATIONS and INEQUALITIES

August 20—26, 1979

Compiled by Károly Lajkó

The Second International Symposium on Functional Equations and Inequalities was held in Debrecen, Hungary from August 20 to August 26, 1979. It was sponsored by the Hungarian Academy of Sciences with additional support from the L. Kossuth University of Debrecen and the Technical University for Heavy Industry of Miskolc. The Symposium was organized by the Mathematical Institute of the L. Kossuth University. The Organizing Committee consisted of Profs. J. BALÁZS, Z. DARÓCZY, I. FENYŐ, M. HOSSZÚ, A. KÓSA, K. LAJKÓ (secretary), L. LOSONCZI, K. TANDORI, E. VINCZE.

There was 75 participants from 13 countries.

The meeting was opened on Tuesday, August 21, 10:00 a.m. by Prof. Z. DARÓCZY. After this the scientific programme began. 52 papers were presented. Among the principal topics considered were: classical equations and types of equations, convex functions, inequalities for mean values, differential inequalities, iterative equations. Many interesting applications were discussed in particular to combinatorics, probability, statistics, information theory and number theory.

The programme of the meeting included an excursion to Hortobágy (the Hungarian Plain) and to a Farmer's Cooperative in Balmazújváros.

The full list of participants follows:

AHMAD, I. R. (Salmiyah, Kuwait)
BAJNOV, D. D. (Sofia, Bulgaria)
BALÁZS, J. (Budapest, Hungary)
BARON, K. (Katowice, Poland)
BÉLTEKY, K. (Debrecen, Hungary)
BENZ, W. (Hamburg, GFR)
BILINSKI, S. (Zagreb, Yougoslavia)
BRYDAK, D. (Kraków, Poland and Port Harcourt, Nigeria)
CHOCZEWSKI, B. (Kraków, Poland)
COLLATZ, L. (Hamburg, GFR)
CZERWIK, S. (Katowice, Poland)
DARÓCZY, Z. (Debrecen, Hungary)
FENYŐ, I. (Budapest, Hungary)
FAWZY, TH. (Riyadh, Saudi Arabia)
GER, J. (Katowice, Poland)
GER, R. (Katowice, Poland)

GESZTELYI, E. (Debrecen, Hungary)
GRZASLEWICZ, A. (Kraków, Poland)
GYIRES, B. (Debrecen, Hungary)
HAJÓSY, A. (Budapest, Hungary)
HOSSZÚ, M. (Gödöllő, Hungary)
JANKÓ, B. (Cluj-Napoca, Roumania and Budapest, Hungary)
JÁRAI, A. (Debrecen, Hungary)
JARCZYK, W. (Katowice, Poland)
KAIRIES, H. H. (Clausthal, GFR)
KÁNTOR, S. (Debrecen, Hungary)
MRS. KÁNTOR (Debrecen, Hungary)
KÁTAI, I. (Budapest, Hungary)
KOMINEK, Z. (Katowice, Poland)
KÓSA, A. (Budapest, Hungary)
KRÄUTER, A. R. (Graz, Austria)
KRÜPPEL, M. (Güstrow, GDR)
KUCZMA, MARCIN E. (Warsaw, Poland)
KUCZMA, MAREK (Katowice, Poland)
LAJKÓ, K. (Debrecen, Hungary)
LOSONCZI, L. (Debrecen, Hungary)
MAKAI, I. (Debrecen, Hungary)
MAKSA, GY. (Debrecen, Hungary)
MRS. MAKSA (Debrecen, Hungary)
MATKOWSKI, J. (Bielsko-Biala, Poland)
MIDURA, S. (Rzeszow, Poland)
MILUSEVA, S. D. (Sofia, Bulgaria)
MOÓR, A. (Sopron, Hungary)
MUNEER, J. (Khartoum, Sudan)
NAGY, B. (Budapest, Hungary)
MRS. NAGY (Budapest, Hungary)
NIKODEM, K. (Katowice, Poland)
NIKODÉMUSZ, A. (Miskolc, Hungary)
PÁLES, ZS. (Debrecen, Hungary)
PELJUH, P. G. (Kiev, USSR)
POWASKA, Z. (Kraków, Poland)
RAISZ, P. (Miskolc, Hungary)
MRS. RAISZ (Miskolc, Hungary)
REICH, L. (Graz, Austria)
RIMÁN, J. (Debrecen, Hungary)
RUTKOVSZKY, E. (Debrecen, Hungary)
SABLIK, M. (Katowice, Poland)
SARKOVSZKIJ, A. N. (Kiev, USSR)
MRS. SCHMAUSER (Miskolc, Hungary)
SCHWAIGER, J. (Graz, Austria)
SMAJDOR, A. (Katowice, Poland)
SNOW, DONALD R. (Provo-Utah, USA)
STERN, M. (Halle, GDR)
SÜMEGI, L. (Debrecen, Hungary)

SZABÓ, GY. (Debrecen, Hungary)
SZÁZ, Á. (Debrecen, Hungary)
SZÉKELYHIDI, L. (Debrecen, Hungary)
TURDZA, E. (Kraków, Poland)
UHRIN, B. (Budapest, Hungary)
V NCZE, E. (Miskolc, Hungary)
WALTER, W. (Karlsruhe, GFR)
WEGRZYK, R. (Warsaw, Poland)
WOŁODŹKO, S. (Kraków, Poland)
ZAHARIEV, A. I. (Sofia, Bulgaria)
ZDUN, M. C. (Katowice, Poland)

During the last session Prof. M. HOSSZÚ thanked the sponsors hosts and participants for their cooperation and effort.

Professors S. BILINSKI and L. COLLATZ thanked, in the name of the participants, the organizing committee and their co-workers to whose efforts both the scientific and social success of the symposium was due.

The conference closed at 12:00 a.m., on Saturday, August 25, 1979.

The abstracts of the talks follow in chronological order of presentation.

NIKODEM, K.: *On quadratic stochastic processes*

Let $(\Omega, \mathscr{A}, P)$ be an arbitrary probability space. A stochastic process $x\colon \mathbf{R}\times\Omega\to\mathbf{R}$ is called:

1) quadratic iff for all $u, v\in\mathbf{R}$

$$x(u+v,\cdot)+x(u-v,\cdot) = 2x(u,\cdot)+2x(v,\cdot) \text{ (a.e.)},$$

2) P-bounded on a non-empty set $A\subset\mathbf{R}$ iff

$$\lim_{n\to\infty}\sup_{t\in A}\{P(\{\omega\in\Omega\colon |x(t,\omega)|\geqq n\})\}=0,$$

3) continuous at the point $t_0\in\mathbf{R}$ iff

$$P\text{-}\lim_{t\to t_0} x(t,\cdot) = x(t_0,\cdot),$$

where P-lim denote the limit in probability.

We have the following

Theorem. *If a stochastic process x: $\mathbf{R}\times\Omega\to\mathbf{R}$ is quadratic then the following conditions are equivalent:*

a) *x is P-bounded on a set $A\subset\mathbf{R}$ of positive inner Lebesgue measure or of the second category with the Baire property,*
b) *x is continuous at every point $t\in\mathbf{R}$,*
c) *for any $t\in\mathbf{R}$ $x(t,\cdot)=t^2x(1,\cdot)$ (a.e.).*

LOSONCZI, L.: *Remarks on additive functions*

Let $\Gamma_n=\{(p_1,\dots,p_n)|p_i\geqq 0,\ \sum_{i=1}^n p_i=1\}$ be the set of all n-ary probability distributions. The general solution of the functional equation

$$\sum_{i=1}^{k} \sum_{j=1}^{l} F(p_i, q_j) = 0 \tag{1}$$

is given, where $F\colon [0, 1]\times[0, 1]\to\mathbf{R}$, $(p_1, \ldots, p_k)\in\Gamma_k$, $(q_1, \ldots, q_l)\in\Gamma_l$, k, l are fixed. Special cases of (1) are also investigated.

CZERWIK, S.: *Random solutions of system of functional equation*

We assume the following hypotheses:
(I) Let (Ω, U, μ) be a probability space. Let $F_i\colon \Omega\times\langle a, b\rangle\times\mathbf{R}^n\to\mathbf{R}$, $(i=1, \ldots, n)$ be measurable in ω, continuous in the second variable and

$$|F_i(\omega, x, y_1, \ldots, y_n)-F_i(\omega, x, z_1, \ldots, z_n)| \leqq \sum_{k=1}^{n} a_{i,k}(\omega)|y_k-z_k|,$$

$$\omega\in\Omega, \quad x\in\langle a, b\rangle, \quad y_i, z_i\in\mathbf{R}, \quad i = 1, \ldots, n,$$

where $a_{i,k}\colon \Omega\to\langle 0, \infty)$, $i, k=1, \ldots, n$.
(II) The functions $f_{i,k}\colon \langle a, b\rangle\to\langle a, b\rangle$, $i, k=1, \ldots, n$ are continuous.

Theorem. *Let* (I), (II) *be fulfilled. If, moreover, the characteristic roots of the matrix* $[a_{i,k}(\omega)]_{i,k=1}^n$ *for* $\omega\in\Omega$ *are by modulus less than one, then the system of equations*

$$F_i(\omega, x, \varphi_1[f_{i,1}(x)], \ldots, \quad \varphi_n[f_{i,n}(x)]) = \varphi_i(x),$$

$$\omega\in\Omega, \quad x\in\langle a, b\rangle, \quad i = 1, \ldots, n$$

has exactly one Carathéodory solution, i.e. there is exactly one system of functions $\varphi_i\colon \Omega\times\langle a, b\rangle\to\mathbf{R}$, $i=1, \ldots, n$ *such that*
1) φ_i, $i=1, \ldots, n$ *are measurable in* ω *and continuous in the second variable,*
2) *for* $\omega\in\Omega$, $x\in\langle a, b\rangle$, $i=1, \ldots, n$

$$F_i(\omega, x, \varphi_1[\omega, f_{i,1}(x)], \ldots, \quad \varphi_n[\omega, f_{i,n}(x)]) = \varphi_i(\omega, x).$$

It is a consequence of a fixed point theorem for the system of random operators and some properties (with respect to measurability) of sets of continuous functions.

KOMINEK, Z.: *On the continuity of additive and Q-convex function*

In the theory of additive and convex functions (in the sense of Jensen) the problem of the continuity is the most important. In particular we may ash about the assumptions on a set T which are sufficient for an additive (or convex) function whose the restriction to the set T is continuous, to be continuous. The following theorems hold:

Theorem 1 ([1]). *If the restriction of an additive real-valued function f of a real variable to an analytic set containing a Hamel basis is continuous, then f is continuous.*

Theorem 2 ([2]). *If the restriction of a real-valued convex function f of a real variable to a set T such that the set*

$$T_1+\ldots+T_n := \{x\in\mathbf{R}|x = t_1+\ldots+t_n;\ t_i\in T = T_i;\ i = 1, \ldots, n\} \tag{1}$$

is a non-open interval is continuous, then f is continuous.

Theorem 3 ([2]). *If the restriction of a real-valued convex function f of a real variable to a second category Baire set is continuous, then f is continuous.*

[1] JONES, F. B., Measure and other properties of a Hamel Basis, *Bull. Amer. Math. Soc.* **48** (1942), 472—481.
[2] KOMINEK, Z., On the continuity of Q-convex functions and additive functions, *Aeq. Math. (to appear).*

WEGRZYK, R.: *On the existence of continuous solutions of functional equations of n-th order with multivalued functions*

We give some theorems about the existence of continuous solutions of the functional equation of n-th order

$$\varphi(x)\in \cdot H(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]), \tag{1}$$

where φ is an unknown function, and a multivalued function H, and functions f_i, $i=1, \ldots, n$ are given. These results can be found in [1].

[1] WEGRZYK, R., Fixed-point theorems for multivalued functions and their applications to functional equations, *Dissert. Math. (to appear).*

KRÜPPEL, M.: *Eindeutigkeitssätze für Funktionalgleichungen*

Für die Funktionalgleichung

$$\varphi[f(x)] = g(x, \varphi(x))$$

wird ein Eindeutigkeitssatz für stetige Lösungen und für die Gleichung

$$\varphi[f(x)] = g[\varphi(x)]$$

ein Eindeutigkeitssatz für streng monotone Lösungen gebracht. Die wesentliche Voraussetzung in beiden Sätzen ist, daß bei der Funktion f keine singulären und regulären Intervalle auftreten (vgl. [1], [2]). Diese Bedingung erfüllen z. B. die Tschebyschew-Polynome in Intervall $[-1, 1]$.

[1] BARNA, B., Über die Iteration reeller Funktionen I, II, III, *Publ. Math. (Debrecen)* 1960, (16—40); 1966, (169—172); 1975, (269—278).
[2] KRÜPPEL, M., Beiträge zur Theorie der vertauschbaren Funktionen, *Math. Nachr.* **56** (1973), 73—100.

TURDZA, E.: *On the inequality $\Psi(x+y) \leqq \Psi(x)\Psi(y)$*

The general and the continuous solutions of the inequality

$$\Psi(x+y) \leqq \Psi(x)\Psi(y), \tag{1}$$

where Ψ: $\mathbf{R} \to \mathbf{R}$, are given.

Let us denote by B — a non-empty additive subgroup of real numbers, by ψ — a function defined and subadditive on B, by A — a set which has exactly one common point with each sum of cosets $[a] \cup [-a]$, $a \notin B$, by $\{\Phi_a\}_{a \in A}$ — a family of functions satisfying the conditions

$$\psi(x+y) \leqq \Phi_a(x) + \Phi_a(y), \quad (x, y) \in [a] \times [-a] \tag{2}$$

$$\Phi_a(x+y) \geqq \Phi_a(x) + \psi(y), \quad (x, y) \in [a] \times B,\ a \notin B. \tag{3}$$

The main result is the following:

The general solution of inequality (1) which takes on positive and negative values has the form

$$\Psi(x) = \begin{cases} e^{\psi(x)}, & x \in B \\ -e^{\Phi_a(x)}, & x \in [a] \cup [-a], \quad a \notin B, \end{cases}$$

where the functions ψ, Φ_a defined above, satisfy conditions (2) and (3).

SZÁZ, Á.: *Quotient spaces defined by linear relations*

Definition. If S is a linear relation from X into Y, then we define

$$Y|S = \{S(x) \colon x \in X\},$$

and

$$S(x) + S(y) = S(x+y) \quad \text{and} \quad \lambda \cdot S(x) = S(\lambda x).$$

Moreover, if in addition X is a topological vector space, then we consider $Y|S$ to be equipped with the finest topology for which the mapping φ_S defined on X by $\varphi_S(x) = S(x)$ is continuous.

Theorem 1. *Let S be a linear relation from X into Y. Then $Y|S$ is a vector space and φ_S is a linear mapping of X onto $Y|S$. Moreover, if in addition X is a topological vector space, then $Y|S$ is also a topological vector space and φ_S is an open mapping.*

Remark. We call $Y|S$ the quotient space of Y defined by S, and φ_S the projection of X onto $Y|S$.

If M is a subspace of a vector space X and R is the linear equivalence relation on X such that $R(0) = M$, then $X|M = X|R$ is called the quotient space of X modulo M.

Theorem 2. *Let S be a linear relation from X onto Y. Then the vector spaces $Y|S$ and $Y|S(0)$ are identical. Moreover, if in addition X is a topological vector space,*

$\mathcal{F}$ is a family of linear selections for S, and Y is equipped with the finest vector topology for which each $f\in\mathcal{F}$ is continuous, then the topological vector spaces $Y|S$ and $Y|S(0)$ are also identical.

Corollary. Let S be a linear relation from a topological vector space X onto Y, $\mathcal{F}$ be a family of linear selections for S, and equip Y with the finest vector topology for which each $f\in\mathcal{F}$ is continuous. Then S is lower semi-continuous.

Theorem 3. *Let S be a linear relation from a topological vector space X onto Y, and equip Y with the coarsest topology for which the projection φ of Y onto $Y|S$ is continuous. Then Y is a topological vector space such that S is an open relation and every linear selection for S is continuous.*

KUCZMA, MARCIN E.: *Differentiation of implicit mappings and Steinhaus' theorem in topological measure spaces*

Considered is the equation of implicit mappings $F(x, y)=z$. Assume that this equation defines each of the three variables as a function of the remaining two: $y=\varphi(x, z)$, $x=\psi(z, y)$; it is well known that if the domain of x, y, z is (a piece of) R^n and if F is C^1 with non-singular $D_x F$, $D_y F$, then also φ and ψ are C^1 and their derivatives are expressed in terms of the derivatives of F. This theorem is extended to the situation where the domain is a topological space with a regular Borel measure; the condition of being of class C^1 is now understood in the sense of the existence and continuity of suitable Radon—Nikodym derivatives.

Steinhaus' theorem which asserts that for any sets A, $B\subset R$ of positive Lebesgue measure the set $A+B$ has interior points admits various generalizations. E.g. (ERDŐS—OXTOBY and others), addition may be replaced by any binary C^1-operation with non-vanishing partial derivatives. Another type of generalization (WEIL) consists in replacing the real line and Lebesgue measure by a locally compact group and Haar measure. A theorem comprising these two results (A, B subsets of a topological measure space, addition replaced by an operation $F(x, y)$ which is coordinate-wise bijective, non-singular and continuously Radon-Nikodym differentiable) is obtained as an application of the extended theorem on the differentiation of implicit mappings.

Problems for further research are indicated.

GRZASLEWICZ, A.: *Remarks to additive functions*

An additive function d: $\mathbf{C}\to\mathbf{C}$ is termed a derivation on $\mathbf{C}$ if it satisfies one of the following conditions

$$d(xy) = x\,d(y)+y\,d(x) \quad (x, y\in\mathbf{C}),$$

$$d(x) = -x^2 d\left(\frac{1}{x}\right) \qquad (x\in\mathbf{C},\ x\neq 0),$$

$$d(x^2) = 2x\,d(x) \qquad (x\in\mathbf{C}).$$

Theorem 1. *If f: $\mathbf{C}\to\mathbf{C}$ is an additive function then the following four conditions are equivalent*

$$f(xy) = xf(x)+yf(x)-xyf(1) \quad (x,\, y\in\mathbf{C}),$$

$$f(x) = -x^2 f\left(\frac{1}{x}\right)+2xf(1) \qquad (x\in\mathbf{C},\, x\neq 0),$$

$$f(x^2) = 2xf(x)-x^2f(1) \qquad (x\in\mathbf{C}),$$

$$f(x) = d(x)+ax \qquad (x\in\mathbf{C}),$$

where d is a derivation on $\mathbf{C}$ and a is a complex constant.

Theorem 2. *An additive function f: $\mathbf{C}\to\mathbf{C}$ satisfies the equation*

$$[f(x)]^2 = x^4\left[f\left(\frac{1}{x}\right)\right]^2 \quad (x\in\mathbf{C},\, x\neq 0)$$

iff $f(x)=ax$ for all $x\in\mathbf{C}$, where a is a complex constant or f is a derivation on $\mathbf{C}$.

Problem: Solve the equation

$$[f(x)]^2 = x^4\left[g\left(\frac{1}{x}\right)\right]^2 \quad (x\in\mathbf{C},\, x\neq 0),$$

where f, g: $\mathbf{C}\to\mathbf{C}$ are additive functions.

Stern, M.: *On the monotonity of a rank function in certain lattices*
Jarczyk, W.: *On a set of functional equations having continuous solutions*

Let $C[0, a]$ denote the space of all real-valued continuous functions defined on a compact interval $[0, a]$. We introduce in $C[0, a]$ the metric

$$\varrho(h_1, h_2) = \sup_{[0,a]} |h_1-h_2|.$$

Let us put

$$C_0[0, a] = \{h\in C[0, a]\colon\ h(0) = 0\}.$$

Suppose that functions $f, g\in C[0, a]$, $0<f(x)<x$ and $g(x)\neq 0$ for every $x\in[0, a]$. We define $H_{f,g}$ as the set of all functions $h\in C[0, a]$ for which the equation

$$\varphi\circ f = g\varphi+h$$

has a solution φ in $C[0, a]$.

Under these assumptions our main result states that if $g(0)=1$, then $H_{f,g}$ is a set of the first category in $C_0[0, a]$, if $g(0)=-1$, then $H_{f,g}$ is a set of the first category in $C[0, a]$. Moreover, if f is a strictly increasing in $[0, a]$ and $|g(0)|\neq 1$, then $H_{f,g}=C[0, a]$.

BILINSKI, S.: *Die zu einer Gruppe gehörenden Funktionalgleichung*

Wenn eine Funktionalgleichung bei Anwendung aller Operatoren ω_v einer diskreten Transformationsgruppe $G_\gamma = \{\omega_1, \ldots, \omega_\gamma\}$ endlicher Ordnung invariant bleibt, so soll diese eine „zur Gruppe G_γ gehörende Funktionalgleichung“ heißen. Es wird die allgemeine Form gegeben in welcher jede zur Gruppe G_γ gehörende Funktionalgleichung dargestellt werden kann, und es werden Beispiele solcher Funktionalgleichungen gegeben und gelöst. Auf diese Beispielen wird auch der Begriff von Paaren konjugirter Funktionalgleichungen erklärt. Man betrachtet auch den Fall, daß die Elemente der Operatorengruppe Substitutionen sind, und auch den besonderen Fall, daß eine solche Gruppe zyklisch ist. Wird die Erzeugende der Gruppe zyklisch und durch elementenfremde Zyklen dargestellt, so wird die Funktionalgleichung, je nach der Anzahl dieser Zyklen, monozyklisch, bizyklisch, trizyklisch, … …, polyzyklisch genannt.

MUNEER JOUSIF ELNOUR: *Hermite Interpolation and Cauchy Problem*

In the following we are going to use Hermite interpolation polynomial for approximating the first and second order differential equations given the boundary conditions. The fundamental points of interpolation are the roots of the Tchebycheff polynomial of the first kind.

We are considering:

1) $$y' = f(x, y), \quad \text{where} \quad f \in C^2([-1, 1] \times \mathbf{R}) \quad \text{and}$$

$$\bar{Q}_{2n+1}(x, y) = \sum_{v=0}^{n+1} \bar{y}_v h_v(x) + \sum_{v=1}^{n} \bar{y}'_v k_v(x).$$

$\bar{y}_v$ and $\bar{y}'_v$ are approximate values obtained by a one step method. $h_v(x)$ and $k_v(x)$ are the modified Hermite fundamental polynomials.

2) $y'' = f(x, y, y')$, where $f \in C^4([-1, 1] \times \mathbf{R}^2)$ and the modified Hermite interpolation polynomial is also given.

In the first case the order of approximation is obtained to be

$$|Q^{(s)}_{2n+1}(x, y) - y^{(s)}(x)| \leqq \frac{A_s \omega_3 \left(\frac{1}{n}\right)}{n^{1-s}}$$

where $s = 0, 1$ and A_s is a constant independent of n and x.

For the second case:

$$|Q^{(s)}_{3n+3}(x, y) - y^{(s)}(x)| \leqq \frac{B_s \omega_6 \left(\frac{1}{n}\right)}{n^{2-s}}$$

where $s = 0, 1, 2$ and B_s is a constant independent of n and x.

DARÓCZY, Z.: *Funktionalgleichungen und Ungleichungen in der Theorie der Mittelwerten*

Es bezeichne $\Omega(\mathbf{R}_+)$ die Menge der differenzierbaren Funktionen φ: $\mathbf{R}_+ \to \mathbf{R}$ mit der Eigenschaft $\varphi'(t) \neq 0$ für alle $t \in \mathbf{R}_+$. Weiter sei $P(\mathbf{R}_+)$ die Menge der stetigen Funktionen f: $\mathbf{R}_+ \to \mathbf{R}_+$. Ist $x=(x_1, \ldots, x_n) \in \mathbf{R}_+^n$, so wird die Größe

$$M\varphi(\underline{x})_f \doteq \varphi^{-1}\Big[\sum_{i=1}^{n} f(x_i)\varphi(x_i) \Big/ \sum_{i=1}^{n} f(x_i)\Big]$$

der Mittelwert von $\underline{x}$ genannt, wo $\varphi \in \Omega(\mathbf{R}_+)$ und $f \in P(\mathbf{R}_+)$ gewählt sind. Im Vortrag werden die homogenen Mittelwerten und einige Ungleichungen für homogenen Mittelwerten angegeben.

BAJNOV, D. D.—ZAHARIEV, A. I.: *О некоторых обобщений интегрального неравенства Гронуолла—Беллмана для функций нескольких переменных*

В работе доказано неравенство Гронуолла—Беллмана для функций нескольких переменных. В случае двух переменных, область интегрирования является компактным множеством, которое в общем случае нельзя представить как декартовое произведение интервалов.

BARON, K.: *On the system of functional inequalities in a single variable*

Let S and X be a set and a topological space, respectively. Fix a positive real number m and a $\xi \in X$. Assuming that a function f mapping $S \times X$ into X and functions β_i, $1 \leqq i \leqq m$, mapping subsets of $S^{[0,\infty]^m}$ into the extended reals are given consider the system of functional inequalities

$$(*) \qquad \lambda_i(x) \leqq \beta_i(\lambda \circ f(\cdot, x)), \quad 1 \leqq i \leqq m.$$

Under what conditions will the only solution λ: $X \to [0, \infty)^m$ of $(*)$ which is continuous at ξ and vanishes ξ be the zero function?

An answer to this question is the subject of our talk.

PELJUH, G. P.: *Исследование систем нелинейных функциональных уравнений с особенностями*

Рассматриваются системы нелинейных функциональных уравнений вида

$$t^{-\nu_i} x_i(\lambda t) = \lambda_i x_i(t) + f_i[t, x_1(t), \ldots, x_n(t)], \quad i = 1, \ldots, n,$$

где $\nu_i > 0$, $0 < \lambda < 1$, $\lambda_i \neq 0$, функций $f_i(t, x_1, \ldots, x_n)$ принадлежат классу C^k (k-целое положительное число, зависящее от ν_i, $i=1, \ldots, n$) в некоторой окрестности начала координат 0.

Дается представление общего решения таких систем при достаточно малых $t > 0$.

Изучается поведение решений при $t \to 0$.

MILUSEVA, SZ. D.—BAJNOV, D. D.: *О применении метода усреднения к одной двухточечной краевой задача для систем дифференциальных уравнений содержащих несколько малых параметров*

В настоящей работе рассматривается двухточечная краевая задача для систем обыкновенных дифференциальных уравнений, содержащая несколько малых параметров с различными порядками малости. Применяется способ получения диагональной усредненной системи, который основывается последовательному применению процесса усреднения по траекториям выражденных систем.

KUCZMA, MAREK: *Discrete limit sets of iterative sequences*

Let X be a topological Hausdorff space and let f: $X \to X$ be a continuous functions. For any $x_0 \in X$ define the iterative sequence $\{x_n\}$ by

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots. \tag{1}$$

Let $L_f(x_0)$ denote the set of the limit points of sequence (1). The problem is discussed as to what can be said about the mapping f on $L_f(x_0)$ if we know that $L_f(x_0)$ is discrete. A particular case is a theorem of B. BARNA, saying that, if X is a compact real interval, and the set $L_f(x_0)$, for a certain $x_0 \in X$, is finite, then $L_f(x_0)$ is a single cycle. We investigate to which extent this theorem can be extended to more general spaces.

MAKSA, GY.: *The general solution of a functional equation related to the mixed theory of information*

Let $\mathbf{R}$ and D denote the set of all real numbers and the set

$$\{(x, y) \in \mathbf{R}^2 \colon x, y \in]0, 1[, \ x+y < 1\},$$

respectively and let α be a fixed positive real number. The functional equation

$$f(1-x)+(1-x)^\alpha f\left(\frac{y}{1-x}\right) = f(y)+(1-y)^\alpha f\left(1-\frac{x}{1-y}\right), \quad (x, y) \in D \tag{1}$$

(where the real-valued unknown function f is defined on $]0, 1[$) has an important role in the mixed theory of information.

KANNAPPAN [3] has shown that (1) implies

$$f(x)+(1-x)^\alpha f\left(\frac{y}{1-x}\right) = f(y)+(1-y)^\alpha f\left(\frac{x}{1-y}\right), \quad (x, y) \in D \tag{2}$$

and

$$f(x) = f(1-x), \quad x \in]0, 1[\tag{3}$$

except if $\alpha=2$. The general solution of the system (2)—(3) is known (see Daróczy [2] and Aczél—Daróczy [1]). In this talk we present the general solution of (1) in case $\alpha=2$.

References

[1] Aczél, J., Daróczy, Z., On measures of information and their characterizations, *Academic Press, New York—London* 1975.
[2] Daróczy, Z., Generalized information functions, *Information and Control*, **16**, 1970.
[3] Kannappan, Pl., Note on generalized information function, *Tohoku Math. J.*, **30**, 1978.

Sablik, M.: *On proper linearly invariant ideals of sets*

In the theory of functional equations in several variables investigated on a restricted domain appears the notion of proper linearly invariant ideal of sets i.e. non empty family I of subsets of the considered group X fulfilling the following conditions

1. $A\cup B\in I$ for every $A, B\in I$,
2. $A\in I,\ B\subset A\Rightarrow B\in I$,
3. $x-A\in I$ for every $x\in X$ and $A\in I$,
4. $X\notin I$.

We give examples for such ideals in topological groups. They are characterized by the property that every set A from the ideal satisfies

$$\lim_{n\to\infty} m(A\cap K_n)/m(K_n) = 0 \quad (\text{or } \liminf_{n\to\infty} m(A\cap K_n)/m(K_n) = 0),$$

where m is a nonnegative and subadditive set function invariant under translations and reflections, while $\{K_n:\ n\in\mathbf{N}\}$ is a sequence of sets satisfying certain conditions. Suitable examples show that these conditions are possibly the weakest ones for which the characterization holds.

Schwaiger, J.: *Analytic Iteration and Infinite-Dimensional Lie Groups*

Given the set $gf(n, \mathbf{C})$ of formal power series $F\in(\mathbf{C}[\![X]\!])^n$ with $F(0)=0$, it is possible to define the structure of a sequential complete locally convex topological $\mathbf{C}$-vector space on it ('convergence of coefficients'). Inspired by papers of D. Pisanelli on similar topics and using the fact, that

$$Gf(n, \mathbf{C}) := \{F\,|\,F\in gf(n, \mathbf{C}),\ F = AX+\mathfrak{P}(X),\ \det A\neq 0\}$$

is a group under substitution of formal power series one is able to put the structure of an infinite dimensional Lie group on $Gf(n, \mathbf{C})$. To do so the theory of analytic functions in (certain) topological vector spaces is used. Applications can be given in the theory of analytic iterations of series $F\in Gf(n, \mathbf{C})$ as developed by L. Reich and J. Schwaiger. It is shown that the analytic iterations of F are exactly the one-parameter groups $(g_\alpha)_{\alpha\in\mathbf{C}}$ of $Gf(n, \mathbf{C})$ taking the value F at $\alpha=1$.

GER, R.: *On a problem of Stanislaw Midura*

In his paper "Sur la détermination de certains sous-groupes du groupe L_s^1 à l'aide d'équations fonctionnelles" (*Dissertationes Mathematicae 105*, *PWN Warszawa*, 1973) S. MIDURA investigates, among others, the functional equation

$$(*) \qquad f\bigl(x+3f(y)^2-3f(x)f(y)-y\bigr)=f(x)-f(y)$$

for functions f mapping the set $\mathbf{R}$ of all real numbers into itself. He proves there that equation $(*)$ has no nontrivial continuous solutions. He quotes also a remark due to Z. MOSZNER that equation $(*)$ admits some discontinuous additive solutions $f\colon \mathbf{R}\to\mathbf{R}$.

During the 17[th] International Meeting on Functional Equations (Oberwolfach, West Germany, June 1979) S. Midura has announced a result establishing the general odd (as well as even) solution of $(*)$; he asked also about the general solution of the equation considered.

This question is surveyed and reported on.

BAJNOV, D. D.—ANGELOV, G. A.: *Глобальные решения нелинейных функциональных уравнений m-ого порядка в пространстве L^p*

При помощи метода аккретивных операторов получены достаточные условия с существовании и единственности глобального решения нелинейных функциональных уравнений в пространстве суммируемых с p-ой степенью Бохнеру функций $(1\leqq p<\infty)$.

BRYDAK, D.: *Monotonic solutions of a nonlinear functional equation*

A sufficient condition for the solutions of the functional equation $\varphi[f(x)]=$ $=g[x, \varphi(x)]$ to be monotonic is given in case where the equation has one-parameter family of continuous solutions. A similar condition is also given for strictly monotonic continuous solutions.

KRÄUTER, A. R.: *Über einen Zusammenhang zwischen fraktioneller und analytischer Iteration bei formal-biholomorphen Abbildungen*

In seiner Arbeit [1], pp. 18—19, Satz 4, hat LUDWIG REICH erstmals das Problem behandelt, unter welchen Voraussetzungen aus der Existenz einer abzählbaren Folge von fraktionellen Iterierten („Wurzeln") einer kontrahierenden biholomorphen Abbildung F die analytische Iterierbarkeit von F folgt (für die entsprechenden Definitionen vergleiche man [1], passim, sowie die dort zitierte Literatur!). Hier soll nun eine Verallgemeinerung dieses Ergebnißes im Falle beliebiger formalbiholomorpher Abbildungen gebracht werden:

Satz.

Voraussetzungen:

(1) *Es sei eine formal-biholomorphe Abbildung, deren Linearteil in JORDANscher Normalform J vorliegt, und es sei* $\Lambda=(\ln \varrho_1, \ldots, \ln \varrho_n)$ *eine beliebige, aber feste Wahl der Logarithmen der Eigenwerte von J.*

(2) *Es existiere eine abzählbare Folge* $(F^{1/r_j})_{j\in \mathbf{N}}$ *von Wurzeln von F mit* $r_j<r_{j+1}$, *so daß der Linearteil* J^{1/r_j} *von* F^{1/r_j} *die Eigenwerte* $\exp\left(\frac{1}{r_j}\ln \varrho_1\right), \ldots, \exp\left(\frac{1}{r_j}\ln \varrho_n\right)$ *besitzt und von kanonischer Struktur ist für alle* $j\in\mathbf{N}$.

(3) *Es seien* F^{1/r_j} *und* $F^{1/r_{j+1}}$ *miteinander vertauschbar für alle* $j\in\mathbf{N}$.

Behauptung:
Dann existiert eine analytische Iteration von F bezüglich Λ.

Literatur:

[1] Ludwig Reich, Über analytische Iteration linearer und kontrahierender biholomorpher Abbildungen. *Berichte der Gesellschaft für Mathematik und Datenverarbeitung* **42** (1971), 1—23.

Reich, L.: *Fraktionelle Iteration formal-biholomorpher Abbildungen und biholomorpher Kontraktionen*

Es sei $F\colon x\mapsto Ax+\mathfrak{p}(x)$ $(x={}^T(x_1, \ldots, x_n)$, A eine nichtsinguläre komplexe (n, n)-Matrix, $\mathfrak{p}(x)={}^T(\mathfrak{p}_1(x), \ldots, \mathfrak{p}_n(x))$ eine Spalte von Potenzreihenvektoren $\mathfrak{p}_j(x)$ mit $\operatorname{ord}\mathfrak{p}_j(x)\geqq 2)$ ein ordnungserhaltender, den Grundkörper $\mathbf{C}$ elementweise festlassender Automorphismus des Potenzreihenringes $\mathbf{C}[\![x_1, \ldots, x_n]\!]$. $\varrho_1, \ldots, \varrho_n$ seien die Eigenwerte von A, $\sigma_j=\varrho_j^{1/m}$, $j=1, \ldots, n$, $m\geqq 2$, seien fest vorgegebene m-te Wurzeln der Eigenwerte. Wir behandeln folgenden

Satz.
F besitzt genau dann eine m-te iterative Wurzel G (kurz $G=F^{1/m}$*), mit den Eigenwerten* $\sigma_1, \ldots, \sigma_n$ *des Linearteils, falls F zu einer Normalform N der folgenden Bauart konjugiert ist: Ist* $N\colon x\to Ix+\mathfrak{N}(x)$, *so treten in der Komponente* $\mathfrak{N}_j(x)$ *von* $\mathfrak{N}(x)$ *höchstens Monome* x^α *wirklich auf, falls die Relation* $\sigma_j=\sigma_1^{\alpha_1}\ldots\sigma_n^{\alpha_n}$ *besteht* $(x^\alpha:=x_1^{\alpha_1}\ldots x_n^{\alpha_n})$.

Anwendungen auf biholomorphe Kontraktionen, Automorphismen ohne m-te Wurzeln etc.

Benz, W.: *On a functional equation of I. Fenyő*

Consider

$$G:=\left\{f\colon \mathbf{R}\to\mathbf{R} \mid f(x)f(y)=f(xy)f\left(\frac{x+y}{2}\right),\ \forall x, y\in\mathbf{R}\right\}$$

and call $F \subset \mathbf{R}$ regular iff

$$\forall x, y \in \mathbf{R} \quad x, y \in F \Leftrightarrow xy, \frac{x+y}{2} \in F.$$

Using regular sets I. Fenyő has constructed functions f in G by setting

$$(*) \qquad f(x) = A\chi_F(x),$$

where A is a real constant, F a regular set, and where χ_F is the characteristic function of F.

Remark. Since $f(x) = \chi_{\mathbf{Q}}(x)$, $\mathbf{Q}$ the field of rationals, is not a solution (because otherwise

$$0 = f(2+\sqrt{3})f(2-\sqrt{3}) = f(1)f(2))$$

a regular set cannot be defined by

$$\forall x, y \in \mathbf{R} \quad x, y \in F \Rightarrow xy, \frac{x+y}{2} \in F.$$

Professor Fenyő posed the problem (s. problem 7., Report on "Conference on functional equations", 1979. Oberwolfach), whether all $f \in G$ are of type $(*)$.

We can prove

Theorem 1. *Given $f \in G$ there exist a regular set F and a real constant A such that*

$$(**) \qquad f(x) = A\chi_F(x), \quad \text{for all real} \quad x \neq 0.$$

Given $f \in G$ define $F_f := \{x \in \mathbf{R} \mid f(x) \neq 0\}$. Consider the two properties
(i) $0 \in F_f$
(ii) $0 \in F_f$ and there exists $a < 0$ in F.

Theorem 2. *Given $f \in G$ such that one of the conditions* (i), (ii) *holds. Then* $(**)$ *is also true for $x = 0$, where $F := F_f$. If none of the conditions* (i), (ii) *holds then*

$$g(x) = \begin{cases} f(x), & x \neq 0 \\ \text{arbitrary}, & x = 0 \end{cases}$$

is also in G.

(For instance

$$g(x) = \begin{cases} 2, & x > 0 \\ 1, & x = 0 \\ 0, & x < 0 \end{cases}$$

is in G.)

Another theorem was mentioned having the consequence, that $f \in G$ is the Heaviside-function iff
(1) $\exists a \in \mathbf{R}\ f(a) = 0,\ \exists 1 \neq b \in \mathbf{R}\ f(b) = 1$,
(2) f is continuous from the right in $x = 0$.

SNOW, DONALD R.: *Using functional equations to obtain Pascal-type properties and triangles for the various combination functions*

This paper shows how Pascal's property (for the binomial coefficients) and generalizations for related functions can be obtained very simply by using a functional equation system which describes the combination functions. These functions are for combinations with various repetitions, e.g. elements used only once (binomial coefficients), twice, at most a given number of times, unlimited repetitions, or various finite numbers of times but not others such as elements used once, twice, or four times, but not three nor more than four times. It will be shown that all of these functions satisfy the same functional equation system but different initial conditions so the study of any of these combination functions is equally simple using this approach. The corresponding Pascal-type property and hence a Pascal-type triangle are obtained immediately, and the approach leads to the generating function and other properties of each function. These generalized Pascal-type triangles satisfy properties analogous to Pascal's triangle itself, such as symmetry in certain cases. The approach shows how all these functions are special cases of one structure and hence gives a means of unifying and studying all related properties of combination functions.

COLLATZ, L.: *Differential-Inequalities in linear and nonlinear Boundary value problems*

Differential Inequalities have been successful in the theory and in the numerical treatment of ordinary and partial differential equations. The numerical treatment uses approximation theory and algorithms of optimization theory, and the use of inequalities is in many cases the only method which gives exact numerical inclusions for the wanted solution. Recently this was done in combination with the Finite-Element methods in conlinear hyperbolic equations and in certain types of free boundary value problems with partial differential equations. Numerical example have been calculated on computers.

FAWZY, TH.: *Matrix inequalities and the approximate solution of the differential equation $y''=f(x, y, y')$ with spline functions*

A new method for approximating solution of the initial value problem $y''=f(x, y', y')$, $y(x_0)=y_0$, $y'(x_0)=y'_0$ is presented. It is a one-step method $O(h^{r+2+\alpha})$ in $y^{(i)}(x)$, $i=0, 1, 2, \dots, r+2$ if $f\in C^r$ and $f\in \mathrm{Lip}\,\alpha$ where $0<\alpha\leqq 1$. The stability of the method and A-stability are also investigated.

SARKOVSZKIJ, A. N.: *Асимптотическое поведение решений нелинейных функциональных уравнений*

Рассматриваются функциональные уравнения вида

$$(*)\qquad F(x(t), x(\varphi(t))) = 0,$$

$x(t)$: $R^+ \to R$, $t < \varphi(t)$ при $t > 0$. Решения таких уравнений, как правило, стремятся к кусочно постоянным функциям или осциллируют с неограниченно растущей частотой колебаний, если $\varphi(t)$ растет не очень быстро, например, если $\varphi(t) \leqq t+h$, $h = \text{const}$. В исключительных случаях уравнения $(*)$ имеют непрерывные периодические решения. В частности, при $\varphi(t) = t+h$ уравнения $(*)$ могут иметь решения не только с периодом h, но и с любым периодом $\neq h$.

LAJKÓ, K.: *Some general functional equations*

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of real and positive real numbers, respectively. f_i: $\mathbf{R} \to \mathbf{R}$ $(i=1, \ldots, n)$, g: $\mathbf{R} \to \mathbf{R}$, G: $\mathbf{R}^{n-1} \to \mathbf{R}$, p_i: $\mathbf{R}_+ \to \mathbf{R}$ $(i=1, \ldots, n)$, q: $\mathbf{R}_+ \to \mathbf{R}$, Q: $\mathbf{R}_+^{n-1} \to \mathbf{R}$ are arbitrary functions.

We give the general solutions of functional equations

$$(1) \qquad \prod_{i=1}^{n} f_i(x_i) = g\left(\sum_{i=1}^{n} x_i\right) G(x_1 - x_n, \ldots, x_{n-1} - x_n), \quad (x_1, \ldots, x_n) \in \mathbf{R}^n,$$

$$(2) \qquad \prod_{i=1}^{n} p_i(x_i) = q\left(\sum_{i=1}^{n} x_i\right) Q\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right), \quad (x_1, \ldots, x_n) \in \mathbf{R}_+^n.$$

The equations (1) and (2) are important in the characterization of normal and gamma distributions, respectively.

FENYŐ, I.: *Solution of a generalized Gamma-function equation*

The holomorphic solution of the functional equation

$$f(z+1) = g(z) f(z) \quad f(1) = 1$$

is studied, where $g(z)$ is a given holomorphic function in $\operatorname{Re} z > 0$.

KÁTAI, I.: *Arithmetical functions*

More then 30 years ago P. ERDŐS proved the following statement: if f is additive, and monotonic or $f(n+1) - f(n) \to 0$ then it is a constant multiple of $\log n$. In 1969 I proved that if f is additive and

$$(1) \qquad \frac{1}{x} \sum_{n \leqq x} |f(n+1) - f(n)| \to 0,$$

then $f = c \log$, that was a conjecture of P. Erdős. Later WIRSING deduced this assuming only that

$$(2) \qquad \begin{cases} \liminf \dfrac{1}{x} \displaystyle\sum_{x \leqq n \leqq (1+\gamma)x} |f(n+1) - f(n)| \to 0 \\ \gamma > 0 \quad \text{const.} \end{cases}$$

In 1976 a great progress has been achieved by E. Wirsing by proving the following statement: If $f(n)$ is complex valued, completely additive and $f(n+1)-f(n)=o(\log n)$, then $f(n)=c\log n$.

Hence I deduced the following assertion. If $\alpha_0, \ldots, \alpha_k$ is a non-trivial sequence of complex numbers, f being a completely additive function such that

$$\frac{\alpha_0 f(n)+\alpha_1 f(n+1)+\ldots+\alpha_k f(n+k)}{\log n} \to 0,$$

then $f\equiv c\log$ if $\alpha_0+\ldots+\alpha_k=0$, while $f\equiv 0$, if $\alpha_0+\ldots+\alpha_k\neq 0$.

Similar assertion has been proved by P.D.T.A. Elliott.

Kairies, H. H.: *Functional equations with Riemann sums*

Let be $f\in L^1[0,1]$ and $D_p(x):=\left|\int\limits_0^1 f(t)\,dt-\frac{1}{p}\sum\limits_{k=0}^{p-1} f\left(\frac{x+k}{p}\right)\right|$. Then the approximation error $D_p(x)$ depends strongly on the function class f belongs to. E.g.

$\lim\limits_{p\to\infty} D_p(x)=0$ uniformly in x, if f is Riemann integrable or $D_p(x)=o\left(\frac{1}{p}\right)$ uniformly in x, if f is of bounded variation over $[0, 1]$. Moreover there is a nice connection between the Fourier coefficients of f and the function $x\mapsto\frac{1}{p}\sum\limits_{k=0}^{p-1} f\left(\frac{x+k}{p}\right)$.

These both facts are used to give existence and uniqueness theorems for functions satisfying functional equations with Riemann sums in certain function spaces.

Those functional equations occur for instance in Nördlund's theory of difference equations: $g(x, y)=\frac{1}{p}\sum\limits_{k=0}^{p-1} g(x+ky, py)$.

(Multiplication theorem of the principal solutions) or in the definition of replicative functions: $a_p g(px)+b_p=\frac{1}{p}\sum\limits_{k=0}^{p-1} g\left(x+\frac{k}{p}\right)$.

Furthermore they are important in the axiomatic theory of many special functions (Γ-function and related functions, Bernoulli polynomials, Hurwitz ζ-function).

Járai, A.: *Regularity properties of functional equations*

In connection with the functional equation

$$f(t) = h(t, y, f_0(y), f_1(g_1(t, y)), \ldots, f_n(g_n(t, y))), \quad (t, y)\in D \tag{1}$$

we prove that if h is a continuous function and f_i is a measurable function on a set A_i with positive measure for $i=1, 2, \ldots, n$, then under certain weak conditions on D, A_i and g_i for $i=1, \ldots, n$, the function f in (1) is continuous. Similar result is proved for functions satisfying the condition of Baire instead of measurability.

GYIRES, B.: *On a matrix valued functional equation*

In the following we are talking about $p \times p$ matrices whose elements are complex numbers and complex valued functions defined on the whole real line respectively.

The matrix valued function $\Phi(t) = (a_{kj}\varphi_{kj}(t))$ is said to be a matrix valued characteristic function on the stochastic matrix $A = (a_{kj})$, if

$$\varphi_{kj}(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_{kj}(x) \quad (k,\, j = 1, \dots, p),$$

where $F_{kj}(x)$ is a probability distribution function.

Let the real numbers $a_k \neq 0$, $b_k \neq 0$, $k = 1, \dots, n$ be given. Let $i_1, \dots, i_n$ be positive integers satisfying the inequality $1 \leqq i_1 < \dots < i_n \leqq 2n$. Let us denote the elements of $1, \dots, 2n$, which are different from $i_1, \dots, i_n$ by $k_1, \dots, k_n$.

Let the matrix valued functional equation

$$(1) \quad \begin{cases} A^{i_1-1}\Phi_{i_1}(a_1t+b_1u)A^{i_2-i_1-1}\Phi_{i_2}(a_2t+b_2u)\dots A^{i_n-i_{n-1}-1}\Phi_{i_n}(a_nt+b_nu)A^{2n-i_n} = \\ = \prod_{k=1}^{2n} \Phi_k(c_k v), \quad t \in \mathbf{R}_1, \quad u \in \mathbf{R}_1 \end{cases}$$

with $c_{i_l}v = a_{i_l}t$, $c_{k_l}v = b_{i_l}u$ $(l = 1, \dots, n)$ be given.

In this paper we deal with the solutions

a) of the functional equation (1) by given $i_1, \dots, i_n$;

b) of the functional equation system (1) if $i_1, \dots, i_n$ runs over all $1 \leqq i_1 < \dots$ $\dots < i_n \leqq 2\mathrm{n}$;

provided that $\Phi_1(t), \dots, \Phi_{2n}(t)$ are matrix valued characteristic functions on A.

The general solution of these two problems are unknown.

The author has solutions in both cases a) and b), if the conditions

$$\Phi_j(t)\Phi_k(u) = \Phi_k(u)\Phi_j(t); \quad t, u \in \mathbf{R}_1 \quad (j, k = 1, \dots, 2n)$$

are satisfied.

CHOCZEWSKI, B.—POWASKA, Z.: *On generalized subadditive functions*

We consider the functional inequality

$$(1) \qquad g(ax+by+c) \leqq G(g(x), g(y))$$

in the case of real functions defined on a linear topological space over R. Functions g satisfying (1) can be regarded as generalized subadditive (or convex) functions. Results concern the form of solutions of (1) (and some of its special cases) and particular properties of these solutions.

HOSSZÚ, M.—HAJÓSY, A.: *Difference equations with connection to Legendre polynomials and some applications in geophysics*

Let $P_n(x)$ be the Legendre polynomial of degree n and

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m}$$

be the associated Legendre polynomials. Considering the expansion

$$P_n^m(x) = \sum_{k=0}^{n} a_{nk}^m \cdot \begin{cases} T_k(x), & \text{if } m \text{ is even} \\ S_k(x), & \text{if } m \text{ is odd} \end{cases}$$

where T_k, S_k are Tchebycheff polynomials, one raises the inverse problem to determine the components α_{nk}^m of T_n and S_n in an expansion by associated Legendre polynomials in form

$$\left.\begin{matrix} T_n(x) \\ S_n(x) \end{matrix}\right\} = \sum_{k=0}^{n} \alpha_{nk}^m P_k^m(x),$$

where we define $P_k^m = T_k$ or S_k for $k<m$. The system of recursions

$$\frac{2k}{2k-1}\alpha_{n,k-1}^0 + \frac{2k+2}{2k+3}\alpha_{n,k+1}^0 = \alpha_{n+1,k}^0 + \alpha_{n-1,k}^0 \tag{1}$$

$$(m-2)(k+m)(k-m+1)\alpha_{nk}^m = (-1)^{m-1}(2m-2)n\alpha_{nk}^{m-1} + m\alpha_{nk}^{m-2}, \tag{2}$$

$$m \geqq 3, \quad n, k \geqq m, \quad n \geqq k$$

can be proved which is equivalent to a system of difference equations of order 2.

Heine in 1842 obtained the explicit form of α_{nk}^0. By the present recurrence all the components can be computed. The method of reduction to difference equations seems to be applicable also for more general orthogonal system. E.g. the same problem for Jacobi polynomials (solved by Feldheim in 1942 by generating functions) can be solved by this elementary method.

MIDURA, S.: *Sur les solutions de l'équation fonctionnelle* $\varphi(\alpha\varphi(\beta)+\beta\varphi^k(\alpha)) = \varphi(\alpha)\varphi(\beta)$

Nous nous proposons d'indiquer certaines propriétés des solutions de l'équation fonctionnelle

$$\varphi(\alpha\varphi(\beta)+\beta\varphi^k(\alpha)) = \varphi(\alpha)\varphi(\beta), \tag{1}$$

où la fonction cherchée φ applique l'ensemble des nombres réels R dans l'ensemble des nombres réels non nuls R_0, k étant un nombre entier non négatif $(\varphi^k(\alpha) = [\varphi(\alpha)]^k)$.

L'équation (1) s'est présentée dans le travail [4], où il s'agissait de déterminer certains sous-demigroupes du groupe L_2^1. Dans le cas où $k=0$ l'équation (1) prend la forme

$$\varphi(\alpha\varphi(\beta)+\beta) = \varphi(\alpha)\varphi(\beta). \tag{1'}$$

L'équation (1') aété étudiée pour la première fois dans le cas où $\varphi\colon R \to R$ par S. GOŁĄB et A. SCHINZEL [2], qui ont déterminé, entre autres, les solutions continues de cette équation. Z. DARÓCZY [1] a trouvé les solutions continues de l'équation (1′) dans le cas où φ est définie sur un espace de Hilbert réel. La solution générale de l'équation (1′) aété donnée par S. WOŁODŹKO [5]; certaines modifications de sa construction ont été présentées par P. JÁVOR [3].

[1] DARÓCZY Z., Az $f[x+yf(x)]=f(x)f(y)$ függvényegyenlet folytonos megoldásairól Hilbert-terekben, *Matematikai Lapok* **17** (1966), 339—343.
[2] GOŁĄB S., SCHINZEL A., Sur l'équation fonctionnelle $f(x+yf(x))=f(x)f(y)$, *Publ. Math. (Debrecen)* **6** (1959), 113—125.

VINCZE, E.: *Über zwei Klassen der Lösungen der Cauchyschen Funktionalgleichung*

UHRIN, B.—DANCS, I.: *A Measure Theoretic Inequality*

For $x, \alpha\in\mathbf{R}^n_+$ and $A\subset\mathbf{R}^n_+$ denote $x^\alpha=(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$, $\frac{1}{\alpha}=\left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right)$, $A^\alpha=\{x^\alpha|x\in A\}$. For $A, B\subset\mathbf{R}^n_+$, $A+B$ means the Minkowski (or algebraic) sum of A and B. The Lebesgue measure in $\mathbf{R}^n$ is denoted by μ. Then, the inequality

$$\mu\left((\lambda A^\alpha+(1-\lambda)B^\alpha)^{\frac{1}{\alpha}}\right)\geqq\left(\lambda\mu(A)^{\varphi(\alpha)}+(1-\lambda)\mu(B)^{\varphi(\alpha)}\right)^{\frac{1}{\varphi(\alpha)}} \tag{1}$$

holds, where $\varphi(\alpha)=\left(\sum_{i=1}^n\alpha_i^{-1}\right)^{-1}$, $0\leqq\lambda\leqq1$, $0\leqq\alpha_i\leqq1$, $i=1, \ldots, n$, and $A, B\subset\mathbf{R}^n_+$ are arbitrary L-measurable sets of positive measure such that $(\lambda A^\alpha+(1-\lambda)B^\alpha)^{\frac{1}{\alpha}}$ is L-measurable.

The inequality (1) contains the classical Brunn—Minkowski inequality as a special case (take $\alpha_i=1$, $i=1, \ldots, n$). In fact, the inequality (1) is the simplest special case of a more general integral inequality for functions. In the paper this more general inequality is proved.

MILUSEVA, SZ. D.—BAJNOV, D. D.: *Обоснование метода усреднения для одного класса дифференциально-функциональных уравнений с импульсами*

В работе обоснован метод усреднения для дифференциально-функциональных уравнений с импульсами и с преобразованным аргументом, являющийся функцией независимого переменного и неизвестной функции.

BAJNOV, D. D.—HRISZTOVA, SZ. G.: *О некоторых интегральных неравенств для функций нескольких переменных*

Рассматриваются нелинейные интегральные неравенства типа Бихари для скалярной функции векторного аргумента.

MATKOWSKI, J.: *On the iterations of some entire functions*

POWASKA, Z.: *Über differenzierbare Lösungen einer Funktionalgleichung*

In diesem Vortrag überlegen wir die Funktionalgleichung

$$\psi(x+y) \leqq F(\psi(x), \varphi(y)) \tag{1}$$

und die Funktionalgleichung

$$\varphi(x+y) = F(\varphi(x), \varphi(y)), \tag{2}$$

wo x, y reelle Zahlen sind, ψ: $\mathbf{R} \to I$ und φ: $\mathbf{R} \to I$ sind die unbekannte Funktionen und die Funktion F: $I \times I \to I$ ist gegeben. I ist ein Interval von reelle Zahlen.
Wir charakterisieren die differenzierbare Lösungen von (1).

SZÉKELYHIDI, L.: *Almost periodic functions and functional equations*

Let G be a topological group. A bounded complex valued function on G is said to be almost periodic if the set of its left translates is relatively compact in the Banach space of all bounded complex valued functions on G. Let us consider the functional equation

$$f(xy) = \sum_{i=1}^{n} g_i(x) h_i(y) \tag{1}$$

on G where the unknown functions f, g_i, h_i are bounded. One of our characteristic result is that f is almost periodic. Similar results can be obtained for more general equations.

In the case of commutative groups using Fourier transformation all bounded solutions of (1) and similar equations can be obtained.

SMAJDOR, A.: *On iterations of analytic functions*

Let E be a complex Banach space. We assume that f is an analytic function in an open set $G \subset E$, $0 \in G$, $f(0)=0$ and $s:=f'(0)$ is an invertible mapping of the space E onto E with $\|s\|^2 \|s^{-1}\| < 1$.

Under above conditions W. Smajdor proved that the sequence $\{s^{-n} f^n(x)\}$ is uniformly convergent in some neighbourhood of zero to an analytic solution of the Schröder equation

$$\sigma[f(x)] = s\sigma(x). \tag{1}$$

This solution is invertible in a neighbourhood of zero.

Definition. A continuous iteration group $\{f^u\}$ of f is called an A-iteration group of f iff all functions $f^u(x)$ are analytic with respect to x.

Theorem 1. *If* $\{f^u\}$ *is an A-iteration group of* f *and* $s^u = (f^u)'(0)$, *then* s^u *is a continuous iteration group of* s *and*

$$f^u(x) = \sigma^{-1}[s^u \sigma(x)], \tag{2}$$

where $\sigma(x) = \lim_{n\to\infty} s^{-n} f^n(x)$.

Conversely, if $\{s^u\}$ is a continuous iteration group of s and functions $f^u(x)$ are defined by (2), then f^u is an A-iteration group of f.

Theorem 2. *Let* M *be a set of positive integers with* $1 \in M$. *If for every* $n \in M$ *there exists an analytic function* f_n *such that*

$$f_n^n(x) = f(x)$$

then there exists an A-iteration group $\{f^u\}$ *of* f *for which the equality*

$$f_n(x) = f^{\frac{1}{n}}(x)$$

holds if and only if there exists a continuous iteration group s^u *of* s *such that* $s^{\frac{1}{n}} = s_n$ *for every* $n \in M$, *where* $s_n = f_n'(0)$.

Theorem 2 gives the solution of a problem raised by Professor LUDWIG REICH in 1973.

ZDUN, M. C.: *Differentiable iteration semigroups*

Let $\{f^t,\ t>0\}$ be an iteration semigroup (i.s.) of continuous functions in a closed interval I. Assume that for every $x \in I$ the mapping $t \to f^t(x)$ is measurable. Put $x_1 := \inf \bigcup_{t>0} f^t[I]$, $x_2 := \sup \bigcup_{t>0} f^t[I]$. Every function f^t has the following properties:

$$\text{(H)} \qquad \begin{cases} f^t(x_1) \geqq x_1, \quad f^t(x_2) \leqq x_2, \\ f^t \text{ has at } x_1 \text{ the absolute min and at } x_2 \text{ the abs. max,} \\ f^t \text{ is increasing in } \langle x_1, x_2\rangle, \text{ but } f^t \text{ can be constant} \\ \text{only in a neighbourhood of its fixed points.} \end{cases}$$

Put $A_f := \{x \mid f(x) = x\}$.

Definition. An i.s. $\{f^t,\ t>0\}$ of f (i.e. $f^1 = f$) is said to be of class $C^1(C_0^1)$ if all functions f^t are of class C^1 in the set $I(I \setminus A_f)$.

Further assume that f is strictly increasing in $\langle x_1, x_2\rangle$.

Theorem 1. *If a function* f *of class* C^1 *satisfies condition* (H) *with some points* $\bar{x}_1, \bar{x}_2, f'(x) \neq 0$ *for* $x \in (\bar{x}_1, \bar{x}_2) \cup A_f$ *and* $f|_{\langle \bar{x}_1, \bar{x}_2\rangle}$ *has an i.s.* $\{g^t,\ t>0\}$ *of class* C^1, *then there exists exactly one i.s.* $\{f^t,\ t>0\}$ *of class* C^1 *of* f *such that* $f^t|_{\langle \bar{x}_1, \bar{x}_2\rangle} = g^t$.

Theorem 2. *Let a function* $f\colon \langle a, b\rangle \to \langle a, b\rangle$ *of class* C^1 *be strictly increasing. If there exists a continuous solution of the functional equation*

$$\varphi[f(x)] = f'(x)\varphi(x) \quad \textit{for} \quad x \in (a, b) \cup A_f$$

different from zero in $(a, b) \setminus A_f$ *and differentiable in* A_f *such that* $\varphi'(x) \neq 0$ *for* $x \in A_f$, *then* f *has an i.s. of class* C^1 *in* $\langle a, b \rangle$.

Some other sufficient conditions for the existence and uniqueness of i.s. of class C^1 of a given function will be given too.

PÁLES, ZS.: *Means defined on a linear normed space*

Let X be a linear normed space and $C \subset X$ be a convex set.

A function E: $C \times C \to X$ is called a differentiable deviation on C if the following conditions are satisfied

(1) $E(x, y) = 0$ if and only if $x = y$,

(2) for every fixed $x \in C$ $y \mapsto E(x, y)$ is a differentiable mapping and $\frac{\partial}{\partial y} E(y, y)$ is an invertible linear operator,

(3) for every $\underline{x} = (x_1, \ldots, x_n) \in C^n$ the set

$$\mathfrak{M}_E(\underline{x}) = \left\{ y \in [\underline{x}] \,\middle|\, \sum_{i=1}^{n} E(x_i, y) = 0 \right\}$$

is not empty. ($[\underline{x}]$ denotes the convex hull of $x_1, \ldots, x_n$.) The set $\mathfrak{M}_E(\underline{x})$ is called a deviation mean of $x_1, \ldots, x_n$.

Let $K \subset X$ be a closed convex cone such that $x, -x \in K$ imply $x = 0$. By the help of K a partial ordering can be defined in X: for $x, y \in K$

$$x \leqq y \quad \text{if} \quad y - x \in K.$$

We investigate the following *problems*:

(i) Which functions are differentiable deviations on C?

(ii) Let E, F be differentiable deviations on C. Find conditions for E, F such that the inequality $e \leqq f$ holds, where $e \in \mathfrak{M}_E(x)$, $f \in \mathfrak{M}_F(x)$, $x \in C^n$, $n \in \mathbf{N}$.

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