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Three-variable $*$-identities and ring homomorphisms of operator ideals

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Abstract. The aim of this paper is to consider triple-type homomorphisms involving the operation of adjoint on some operator ideals and to examine the question of how the two equations defining the multiplicativity and the $*$-preserving property of $*$-homomorphisms can be expressed via only one three-variable equation.

Introduction

Triple systems of operators and triple homomorphisms of operator algebras are of great importance in operator theory because of their applications in modern treatments of the mathematical foundations of quantum mechanics [4] as well as in the theory of complex functions in several and an infinite number of variables [12]. Furthermore, similar additive mappings onto prime rings are also extensively studied from the purely ring theoretical point of view (see [2,3]) and the references therein). As in other cases, if the underlying structure carries an involution, then we are attracted to consider such mappings that preserve somehow this important characteristic.

These preliminaries give us the motivation to consider all (seven) possible so-called $*$-identities of order three on algebras of operators acting on a Hilbert space. Here, by a $*$-identity on an involutive algebra $\mathcal{A}$ we mean the following correspondence

$$\phi(\prod_{k=1}^{n} \sigma_k(x_k)) = \prod_{k=1}^{n} \sigma_k(\phi(x_k)) \quad (x_k \in \mathcal{A}),$$

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where $\phi : \mathcal{A} \to \mathcal{A}$ is assumed to be additive while σ_k is either the identity operator or the involution on $\mathcal{A}$. We also require that for at least one k we have $\sigma_k(x) = x^*$ $(x \in \mathcal{A})$. In this paper we are concerned with the case of $n = 3$ and our aim is to determine how close the mappings satisfying any of the corresponding $*$-identities are to $*$-homomorphisms. Observe that every $*$-homomorphism of $\mathcal{A}$ satisfies every $*$-identity but the converse is obviously not true in general.

The three-variable $*$-identities we are interested in can be divided into three types as follows.

(A1) $\phi(x^*yz) = \phi(x)^*\phi(y)\phi(z)$

(A2) $\phi(xy^*z) = \phi(x)\phi(y)^*\phi(z)$

(A3) $\phi(xyz^*) = \phi(x)\phi(y)\phi(z)^*$

(B1) $\phi(x^*y^*z) = \phi(x)^*\phi(y)^*\phi(z)$

(B2) $\phi(x^*yz^*) = \phi(x)^*\phi(y)\phi(z)^*$

(B3) $\phi(xy^*z^*) = \phi(x)\phi(y)^*\phi(z)^*$

(C) $\phi(x^*y^*z^*) = \phi(x)^*\phi(y)^*\phi(z)^*$.

To sum up our results presented below, we can state that the $*$-identities of type B characterize the $*$-homomorphisms up to the multiplication by ± 1, (C) characterizes them up to the multiplication by $\pm 1, \pm i$, (A1), (A3) characterize them up to the multiplication by an arbitrary complex constant of modulus one while the remaining identity (A2) is somewhat exceptional. The mappings satisfying this later identity are considerably more general and can be represented by two unitary parameters. Finally, as a corollary we give the complete description of mappings satisfying any of the two-variable $*$-identities.

The underlying algebras that we work on are the well-known Schatten $\mathcal{C}_p$ classes of compact operators acting on a complex Hilbert space $\mathcal{H}$ (see [6,7]) which are commonly called also as noncommutative l_p spaces.

It is important to emphasize that by a $*$-homomorphism we mean a *ring* homomorphism even when such a mapping acts on an algebra. That is, we do not require that the mappings in question be linear, we assume them to be merely additive. This approach, i.e. to consider an operator algebra only as a ring has a long history going back to the classical papers [1,5]. Important recent results on additive homomorphisms and derivations can be found in [8–11].

In what follows $B(\mathcal{H})$ stands for the algebra of all bounded linear operators acting on $\mathcal{H}$. $F(\mathcal{H}) \subset B(\mathcal{H})$ is the ideal of all finite rank operators.

In the proofs we use several times Brešar's results on triple homomorphisms of rings (see [2,3]). Among other things, he proved [2, Proof of Theorem 3.3] that a triple homomorphism of a ring onto a prime ring with characteristic not 2 is a ring homomorphism or its negative. Recall that a ring $\mathcal{R}$ is called prime if $a\mathcal{R}b = \{0\}$ always implies $a = 0$ or $b = 0$.

We also use our previous partial result from [13]. This was formulated not for the Schatten classes but for some norm closed subspaces of $B(\mathcal{H})$ containing compact operators. Since the Schatten classes are also Banach spaces (although not in the operator norm, but the specific norm is not important in [13]) and contain all finite rank operators, the same proof works for them as well. Therefore, as a corollary of our work in [13], we have

Proposition 1. *Let $\mathcal{A}$ be a Schatten class $\mathcal{C}_p$ of compact operators acting on a Hilbert space $\mathcal{H}$ $(1 \leq p \leq \infty)$ and $\phi : \mathcal{A} \to \mathcal{A}$ a complex-linear, bijective mapping. If ϕ fulfills* (A2), *then there exist unitary operators $u, v \in \mathcal{B}(\mathcal{H})$ such that $\phi(x) = uxv$ for all $x \in \mathcal{A}$.*

1. Identities of type B and C

We begin with the easier $*$-identites of type B and C. Our results can be formulated for the more general case of prime complex normed $*$-algebras with approximate identity as follows.

Theorem 1. *Let $\mathcal{A}$ be a prime normed $*$-algebra with approximate identity. Then all surjective, additive mappings on $\mathcal{A}$ that satisfy any of the $*$-identities of type B are $*$-homomorphisms or their negatives.*

Remark 1. Observe that any prime C^*-algebra and any Schatten class $\mathcal{C}_p$ $(1 \leq p \leq \infty)$ fulfill the above assumption.

Proof. Assume that a surjective, additive mapping $\phi : \mathcal{A} \to \mathcal{A}$ satisfies (B2). Let $a, b \in \mathcal{A}$ be fixed for a moment. We compute $I = \phi((ab)^* y (ab)^*)$ in two different ways. First we have

$$I = \phi(ab)^* \phi(y) \phi(ab)^*. \tag{1}$$

On the other hand, we infer

$$\begin{aligned} I &= \phi(b^*(a^* y b^*) a^*) = \phi(b)^* \phi(a^* y b^*) \phi(a)^* \\ &= \phi(b)^* \phi(a)^* \phi(y) \phi(b)^* \phi(a)^*. \end{aligned} \tag{2}$$

If we compare (1) and (2), then using the surjectivity of ϕ, with $A = \phi(ab)^* - \phi(b)^*\phi(a)^*$ and $B = \phi(ab)^* + \phi(b)^*\phi(a)^*$ we obtain that

$$AwB + BwA = 0 \quad (w \in \mathcal{A}). \tag{3}$$

It is easy to see that in prime rings (3) implies $A = 0$ or $B = 0$ (see [2], Lemma 1.1). Hence, for every $a, b \in \mathcal{A}$ we have either $\phi(ab) = \phi(a)\phi(b)$ or $\phi(ab) = -\phi(a)\phi(b)$.

Remark 2. We note that several applications of the above argument can be found in [2] and [3].

By a rather standard argument, it now follows that ϕ is either a homomorphism or the negative of a homomorphism. Indeed, if $a \in \mathcal{A}$ is fixed, then the sets

$$P_a = \{b \in \mathcal{A} \colon \phi(ab) = \phi(a)\phi(b)\}$$

and

$$Q_a = \{b \in \mathcal{A} \colon \phi(ab) = -\phi(a)\phi(b)\}$$

are additive subgroups of $\mathcal{A}$ whose union is $\mathcal{A}$. Since a group cannot written as the union of two of its proper subgroups, we infer that either $P_a = \mathcal{A}$ or $Q_a = \mathcal{A}$. Using a similar argument, we finally obtain that ϕ is either a homomorphism or the negative of a homomorpism.

It remains to prove that ϕ is $*$-preserving. We may suppose that ϕ is a homomorphism. Then, using the identity (B2), we get

$$\phi(x)^*\phi(y)\phi(z)^* = \phi(x^*yz^*) = \phi(x^*)\phi(y)\phi(z^*).$$

Let $\phi(y)$ run through an approximate identity. We obtain

$$\phi(x^*z^*) = \phi(x^*)\phi(z^*) = \phi(x)^*\phi(z)^*$$

for all $x, z \in \mathcal{A}$. Hence, for $x, y, z \in \mathcal{A}$, we have

$$\begin{aligned}\phi(x)^*\phi(y)\phi(z)^* &= \phi(x^*yz^*) = \phi(x^* \cdot (zy^*)^*)\\ &= \phi(x)^*\phi(zy^*)^* = \phi(x)^*[\phi(z)\phi(y^*)]^* = \phi(x)^*\phi(y^*)^*\phi(z)^*\end{aligned}$$

and, consequently, it follows that $\mathcal{A}(\phi(y) - \phi(y^*)^*)\mathcal{A} = 0$. The existence of an approximate identity clearly implies that $\phi(y^*) = \phi(y)^*$.

Now, assume that ϕ satisfies (B1). Then we have

$$\begin{aligned}\phi(a)^*\phi(b)^*\phi(xyz) &= \phi(a^*b^*xyz) = \phi(a^*(y^*x^*b)^*z)\\ &= \phi(a)^*(\phi(y^*x^*b))^*\phi(z) = \phi(a)^*(\phi(y)^*\phi(x)^*\phi(b))^*\phi(z)\\ &= \phi(a)^*\phi(b)^*\phi(x)\phi(y)\phi(z).\end{aligned}$$

It follows that ϕ is a triple homomorphism. Since $\mathcal{A}$ is prime, by Brešar's result we get that either ϕ or $-\phi$ is a homomorphism. Assume the first possibility. From the equations

$$\phi(x^*)\phi(y^*)\phi(z) = \phi(x^*y^*z) = \phi(x)^*\phi(y)^*\phi(z)$$

we have $\phi(x^*)\phi(y^*) = \phi(x)^*\phi(y)^*$. Finally, we obtain

$$\begin{aligned}\phi(x)^*\phi(y)^*\phi(z^*)^* &= \phi(yx)^*\phi(z^*)^* = \phi((yx)^*z^{**})\\ &= \phi(x^*y^*z) = \phi(x)^*\phi(y)^*\phi(z).\end{aligned}$$

Apparently, this implies that ϕ is *-preserving. If ϕ satisfies (B3), we can argue in a similar way. □

Theorem 2. *Let $\mathcal{A}$ be a prime complex normed $*$-algebra with approximate identity. Suppose that $a^*a + b^*b = 0$ implies $a = b = 0$ for all $a, b \in \mathcal{A}$. Then all surjective, additive mappings fulfilling the $*$-identity (C) are $*$-homomorphisms multiplied by one of the following numbers: $1, -1, i, -i$.*

Remark 3. Observe that every prime C^*-algebra and any Schatten class obviously satisfies the above assumptions.

Proof. We divide the proof into several steps.

Step 1. Let $\mathcal{R}$ be a prime ring of characteristic not 2 and $A, B \in \mathcal{R}$. If $AwA = BwB$ for all $w \in \mathcal{R}$, then $A = \pm B$.

One can easily see that we have $(A+B)w(A-B)+(A-B)w(A+B) = 0$ for every $w \in \mathcal{R}$. Now, [2, Lemma 1.1] applies.

Step 2. Let ϕ be a surjective, additive mapping satisfying (C). Then $\phi(abc) = \phi(a)\phi(b^*)^*\phi(c)$ for all $a, b, c \in \mathcal{A}$.

Compute $I = \phi((abc)^*x^*(abc)^*)$ in two ways. First we have

$$I = \phi(abc)^*\phi(x)^*\phi(abc)^*$$

and secondly

$$\begin{aligned}I &= \phi(c^*(bcxab)^*a^*) = \phi(c)^*\phi(bcxab)^*\phi(a)^*\\ &= \phi(c)^*\phi(b^{**}(a^*x^*c^*)^*b^{**})^*\phi(a)^* = \phi(c)^*(\phi(b^*)^*\phi(a^*x^*c^*)^*\phi(b^*)^*)^*\phi(a)^*\\ &= \phi(c)^*\phi(b^*)\phi(a)^*\phi(x)^*\phi(c)^*\phi(b^*)\phi(a)^*.\end{aligned}$$

Since ϕ is surjective, it follows from Step 1 that for every $a, b, c \in \mathcal{A}$ we have either $\phi(abc) = \phi(a)\phi(b^*)^*\phi(c)$ or $\phi(abc) = -\phi(a)\phi(b^*)^*\phi(c)$. Now, by the argument that has been followed after Remark 2 above, we infer that either $\phi(abc) = \phi(a)\phi(b^*)^*\phi(c)$ or $\phi(abc) = -\phi(a)\phi(b^*)^*\phi(c)$ holds for all $a, b, c \in \mathcal{A}$.

We are going to show that the second possibility cannot occur. Since

$$\phi(x)^*\phi(y)^*\phi(z)^* = \phi(x^*y^*z^*) = -\phi(x^*)\phi(y)^*\phi(z^*),$$

the surjectivity combined with the existence of an approximate identity implies that

$$\phi(x)^*\phi(z)^* = -\phi(x^*)\phi(z^*) \quad (x, z \in \mathcal{A}). \tag{4}$$

From (C) and (4) we get

$$\phi(x^*y^*z^*) = \phi(x)^*\phi(y)^*\phi(z)^* = -\phi(x)^*\phi(y^*)\phi(z^*),$$

which further implies that

$$\begin{aligned}\phi(ab)^*\phi(x)\phi(ab) &= -\phi((ab)^*xab) = -\phi(b^*(a^*xa)b)\\ &= \phi(b)^*\phi(a^*xa)\phi(b) = -\phi(b)^*\phi(a)^*\phi(x)\phi(a)\phi(b).\end{aligned}$$

Using the existence of an approximate identity again, we obtain

$$\phi(ab)^*\phi(ab) + [\phi(a)\phi(b)]^*[\phi(a)\phi(b)] = 0$$

and so it follows that $\phi = 0$, which is a contradiction.

Step 3. We have $\phi = \lambda\psi$, where $\lambda \in \{1, -1, i, -i\}$ and ψ is a $*$-homomorphism.

First, using Step 2 and (C), we have

$$\begin{aligned}\phi(x_1x_2x_3x_4x_5) &= \phi(x_1)\phi(x_4^*x_3^*x_2^*)^*\phi(x_5)\\ &= \phi(x_1)(\phi(x_4)^*\phi(x_3)^*\phi(x_2)^*)^*\phi(x_5)\\ &= \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x_5).\end{aligned}$$

Next we compute

$$\begin{aligned}\phi(xyz)\phi(w)^*\phi(xyz) &= \phi(xyz)\phi(w^{**})^*\phi(xyz)\\ &= \phi(xyzw^*xyz) = \phi(x)\phi(y)\phi(zw^*x)\phi(y)\phi(z)\\ &= \phi(x)\phi(y)\phi(z)\phi(w)^*\phi(x)\phi(y)\phi(z).\end{aligned}$$

Just as above, using Step 1 we easily obtain that ϕ is either a triple homomorphism or a negative of a triple homomorphism. In the first case Brešar's result gives us that ϕ is either a homomorphism or its negative. If ϕ is a homomorphism, then by Step 2 we have

$$\phi(x)\phi(y^*)^*\phi(z) = \phi(xyz) = \phi(x)\phi(y)\phi(z)$$

and this obviously implies that ϕ is $*$-preserving. If ϕ is the negative of a homomorphism, we can argue in a similar way. In the second case $i\phi$ is a triple homomorphism and, consequently, $i\phi$ or $-i\phi$ is a $*$-homomorphim. □

2. Identities of type A

Theorem 3 below is to be considered the main result of the paper. It asserts that in the case of three variables, (A2) is the most general identity.

First few notes about the notation. If a is a bounded linear operator on a Hilbert space $\mathcal{H}$, then we denote by a^T its transpose with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$. This is not to be confused by the adjoint a^*. The operation $a \mapsto a^T$ is linear (not conjugate-linear) and satisfies $(ab)^T = b^T a^T$. These operations commute, i.e. $a^{*T} = a^{T*}$. We denote by $\langle\ ,\ \rangle$ the inner product in $\mathcal{H}$, $\varrho^\perp$ stands for the orthogonal complement of the subset ϱ and $\alpha\otimes\beta$ denotes the rank-one operator defined by $(\alpha \otimes \beta)\xi = \langle \xi, \beta\rangle\alpha$ $(\xi \in \mathcal{H})$.

Theorem 3. *Let $\mathcal{A}$ be either a Schatten class of compact operators acting on the separable Hilbert space $\mathcal{H}$ or let $\mathcal{A} = B(\mathcal{H})$. Then every surjective, additive mapping that satisfies one of the $*$-identities* (A1) *and* (A3) *is a $*$-homomorphism multiplied by a complex constant of modulus one. Moreover, every surjective and additive mapping ϕ fulfilling the $*$-identity* (A2) *is of the form*

$$(i) \qquad \phi(x) = uxv$$

or

$$(ii) \qquad \phi(x) = u(x^{*T})v$$

where u, v are unitary operators acting on $\mathcal{H}$.

Proof. We divide the proof into several steps again.

Step 1. Each surjective, additive mapping ϕ satisfying (A1) or (A3) also fulfills (A2).

Suppose that (A1) holds for ϕ. We compute

$$\begin{aligned}\phi(abxcd) &= \phi(x^*b^*a^*)^*\phi(c)\phi(d)\\ &= (\phi(x)^*\phi(b^*)\phi(a^*))^*\phi(c)\phi(d) = \phi(a^*)^*\phi(b^*)^*\phi(x)\phi(c)\phi(d).\end{aligned}$$

On the other hand we infer

$$\phi(abxcd) = \phi(a^*)^*\phi(b)\phi(xcd) = \phi(a^*)^*\phi(b)\phi(x^*)^*\phi(c)\phi(d).$$

Since ϕ is surjective, it follows that $\phi(b^*)^*\phi(x) = \phi(b)\phi(x^*)^*$ holds true for all $b, x \in \mathcal{A}$. Therefore, we have

$$\phi(xy^*z) = \phi(x^*)^*\phi(y^*)\phi(z) = \phi(x)\phi(y)^*\phi(z).$$

Now, one can treat (A3) in a similar way. Thus, from now on we suppose that ϕ fulfills (A2).

Step 2. Let $p \in B(\mathcal{H})$ be a minimal (i.e. a rank-one) projection and $x \in B(\mathcal{H})$. Suppose that for every minimal projection q, the condition $pq = qp = 0$ implies $qx = xq = 0$. Then $x \in \mathbb{C}p$.

There exists a unit vector $\xi \in \mathcal{H}$ such that $p = \xi \otimes \xi$. Let $q = \eta \otimes \eta$ with $\|\eta\| = 1$ and $\langle \xi, \eta \rangle = 0$. Hence, $qx = xq = 0$. From $xq = 0$ we obtain $x\eta = 0$ and, similarly, from $qx = 0$ we get $x^*\eta = 0$. Since η was arbitrary, we have $x(\{\xi\}^\perp) = 0$ and $x^*(\{\xi\}^\perp) = 0$. This gives us that $x\xi \perp \{\xi\}^\perp$ which results in $x\xi \in \mathbb{C}\xi$ and the assertion is now obvious.

Step 3. Let $a \in B(\mathcal{H})$ and let a minimal projection $p \in B(\mathcal{H})$ be fixed. If $pya = 0$ for all $y \in F(\mathcal{H})$, then $a = 0$.

The statement would follow from the more general fact that the ring $B(\mathcal{H})$ is prime, but this can also be easily seen as follows. Let ξ be a unit vector such that $p = \xi \otimes \xi$. Then we have $\xi \otimes (a^*y^*\xi) = 0$ and thus $a^*y^*\xi = 0$ holds for all $y \in F(\mathcal{H})$. Since $F(\mathcal{H})\xi = \mathcal{H}$, it follows that $a^* = 0 = a$.

Step 4. Let $p \in B(\mathcal{H})$ be a minimal projection and let $x \in \mathcal{A}$ be such that $\phi(x) = p$. Then one of the following assertions holds true.

(i) $\phi(\lambda x) = \lambda p$ for all $\lambda \in \mathbb{C}$;

(ii) $\phi(\lambda x) = \overline{\lambda} p$ for all $\lambda \in \mathbb{C}$.

Let q be a minimal projection such that $qp = pq = 0$. Plainly, $q \in \mathcal{A}$ and so $q = \phi(y)$ holds true for some $y \in \mathcal{A}$. We have

$$\begin{aligned}\phi(\lambda x)q &= \phi(\lambda x)\phi(y)^*\phi(y) = \phi(\lambda xy^*y)\\ &= \phi(x)\phi(y)^*\phi(\lambda y) = pq\phi(\lambda y) = 0.\end{aligned}$$

We can verify $q\phi(\lambda x) = 0$ in a similar way. Hence, by Step 2, we have $\phi(\lambda x) \in \mathbb{C}p$ and we can define an additive function $\tau : \mathbb{C} \to \mathbb{C}$ by $\phi(\lambda x) = \tau(\lambda)p$. Suppose first that $\lambda \in \mathbb{R}$. We can compute

$$\tau(\lambda)p = \phi(\lambda xx^*x) = \phi(x(\lambda x)^*x) = \overline{\tau(\lambda)}p$$

and this implies that $\tau(\lambda) \in \mathbb{R}$. Furthermore, for every $\lambda, \mu \in \mathbb{R}$ we have

$$\tau(\lambda\mu)p = \phi(\lambda\mu x x^* x) = \phi(\lambda x x^* \mu x) = \tau(\lambda)\tau(\mu)p$$

and this shows that τ is a nonzero ring endomorphism of $\mathbb{R}$. Using a classical result from the theory of functional equations we infer $\tau(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$. In a similar way we can verify that $-1 = \tau(i^2) = \tau(i)^2$, i.e. $\tau(i) \in \{i, -i\}$ which, combined with the additivity of τ, yields the result.

Step 5. ϕ is either linear or conjugate-linear.

Take a minimal projection $p \in \mathcal{A}$ and let $x \in \mathcal{A}$ be such that $\phi(x) = p$. Suppose that ϕ is linear on $\mathbb{C}x$. Pick an $y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. We prove that $\phi(\lambda y) = \lambda\phi(y)$ $(\lambda \in \mathbb{C})$. Let $z \in \mathcal{A}$ and write $z^* = \phi(u)$. Then we have

$$\begin{aligned} pz\phi(\lambda y) &= \phi(x)\phi(u)^*\phi(\lambda y) = \phi(\lambda x u^* y) \\ &= \phi(\lambda x)\phi(u)^*\phi(y) = \lambda pz\phi(y). \end{aligned}$$

By Step 3, $\phi(\lambda y) = \lambda\phi(y)$ follows. If ϕ is conjugate-linear on $\mathbb{C}x$, then one can argue in a similar way.

Step 6. If ϕ is not injective, then Ker ϕ contains all finite rank operators.

Let $a \in$ Ker ϕ and suppose $a \neq 0$. Then $\xi_0 = a\eta_0 \neq 0$ for some $\xi_0, \eta_0 \in \mathcal{H}$. It is sufficient to prove that every operator $\xi \otimes \eta$ lies in Ker ϕ. Since $\xi \otimes \eta = b(\xi_0 \otimes \eta_0)^* c$, where $b = \frac{1}{\|\eta_0\|^2}\xi \otimes \eta_0$ and $c = \frac{1}{\|\xi_0\|^2}\xi_0 \otimes \eta$, hence

$$\phi(\xi \otimes \eta) = \phi(b)\phi(\xi_0 \otimes \eta_0)^*\phi(c).$$

On the other hand, we infer

$$a(\xi_0 \otimes \eta_0)^* \frac{\xi_0 \otimes \eta_0}{\|\xi_0\|^2} = \xi_0 \otimes \eta_0.$$

Consequently, we have

$$\phi(\xi_0 \otimes \eta_0) = \phi(a)\phi(\xi_0 \otimes \eta_0)^*\phi(\frac{\xi_0 \otimes \eta_0}{\|\xi_0\|^2}) = 0.$$

This proves our assertion.

Step 7. The mapping ϕ is injective.

Suppose, on the contrary, that it is not injective. Then we obtain $\phi(F(\mathcal{H})) = 0$. Let $x \in \mathcal{A}$ be such that $\phi(x) = p$ is a nonzero projection. Hence

$$\phi(xx^*x) = \phi(x)\phi(x)^*\phi(x) = pp^*p = p$$

which gives $\phi(xx^*x - x) = 0$. For any $y \in \mathcal{A}$ we have

$$\phi(xx^*xx^*y - xx^*y) = \phi((xx^*x - x)x^*y) = \phi(xx^*x - x)\phi(x)^*\phi(y) = 0.$$

With $h = xx^*$ this further implies

$$0 = \phi((h^2 - h)y^*z) = \phi(h^2 - h)\phi(y)^*\phi(z).$$

Since ϕ is surjective, we obtain $\phi(h^2) = \phi(h)$.

We now distinguish two different cases. If $1 \notin \mathrm{sp}(h)$ ($\mathrm{sp}(h)$ stands for the spectrum of h), then there exists an $y \in \mathcal{B}(\mathcal{H})$ such that $(1 - h)y = 1$. Hence $h^3 = h(h - h^2)yh$ and then we have

$$\phi(h^3) = \phi(h)\phi(h - h^2)^*\phi(yh) = 0$$

which yields $\phi(h)\phi(h)^*\phi(h) = 0$. Clearly, it follows that $\phi(h) = 0$. Note that the Schatten classes are ideals of $B(\mathcal{H})$, so $\phi(yh)$ makes sense.

Now, let $1 \in \mathrm{sp}(h)$. We assert that h is compact. In fact, this needs proof only in the case when $\mathcal{A} = B(\mathcal{H})$. So, let us consider this possibility. Using the identity (A2), it is not hard to see that the proper subspace $\phi^{-1}(0)$ of $B(\mathcal{H})$ is an ideal. Using a classical theorem of Calkin, we obtain that $\phi^{-1}(0)$ is included in the ideal of compact operators. Consequently, it follows also in this case that h is compact. We then have $h = p_1 + \sum_{\lambda_n \neq 1} \lambda_n p_n$ where $0 < \lambda_n$ and the p_ns are pairwise orthogonal finite rank projections. Since $\phi(p_1) = 0$, hence for the operator $k = \sum_{\lambda_n \neq 1} \lambda_n p_n$ we obtain that $\phi(h) = \phi(k)$. Clearly, k is positive and its spectrum does not contain 1. Taking the relations $h = p_1 + k$ and $p_1 k = k p_1 = 0$ into consideration, it follows that $h^2 = p_1 + k^2$ and we arrive at $\phi(k^2) = \phi(h^2) = \phi(h) = \phi(k)$. Just as above, we have $\phi(k) = 0$.

Consequently, in both cases we obtain $\phi(xx^*) = 0$. Thus

$$\phi(xx^*xx^*x) = \phi(xx^*)\phi(xx^*)^*\phi(x) = 0.$$

But, on the other hand, we have

$$\phi(xx^*xx^*x) = \phi(x)\phi(xx^*x)^*\phi(x) = \phi(x)\phi(x)^*\phi(x)\phi(x)^*\phi(x) = p^5 = p$$

and this yields the desired contradiction.

Step 8. Our theorem is true in the case of the identity (A2).

If ϕ is linear, we can now use Proposition 1. In case ϕ is conjugate-linear define $\psi : \mathcal{A} \to \mathcal{A}$ by $\psi(x) = \phi(x^{*T})$. It is easy to see that ψ is linear and satisfies (A2). Consequently, ψ is of the form (i) and so ϕ is of the form (ii).

Step 9. Our theorem is true for the identities (A1) and (A3).

If we insert $\phi(x) = uxv$ into (A1), we obtain

$$ux^*yzv = v^*x^*yvuzv \quad (x, y, z \in \mathcal{A}).$$

From this $ux^*y = v^*x^*yvu$ easily follows. Multiplying this equation by v from the left, we obtain that the operator vu commutes with every finite rank operator. It is well-known that this implies $vu = \lambda$ for some $\lambda \in \mathbb{C}$. Therefore, $v = \lambda u^*$ and hence $\phi(x) = \lambda uxu^*$ $(x \in \mathcal{A})$. This completes the proof for the mappings which are of the form (i) and a similar argument applies to the case of (ii). The identity (A3) can be treated in the same fashion. $\square$

From what we have obtained above, it follows that every additive $*$-epimorphism of the Schatten classes as well as that of $B(\mathcal{H})$ is automatically an either linear or conjugate-linear $*$-automorphism. The forms of these automorphisms can also be easily seen from Theorem 3. Finally, combining this with Theorem 1 and Theorem 2, we have the following corollaries.

Corollary 1. *Let $\mathcal{A}$ be a Schatten class of compact operators acting on the separable Hilbert space $\mathcal{H}$ or let $\mathcal{A} = B(\mathcal{H})$. If ϕ is a surjective, additive mapping satisfying the $*$-identity (C), then there exists a unitary operator $u \in B(\mathcal{H})$ and a scalar $\lambda \in \{1, -1, i, -i\}$ such that either $\phi(x) = \lambda uxu^*$ for all $x \in \mathcal{A}$ or $\phi(x) = \lambda u(x^{T*})u^*$ for all $x \in \mathcal{A}$.*

Corollary 2. *Let $\mathcal{A}$ be a Schatten class of compact operators acting on the separable Hilbert space $\mathcal{H}$ or let $\mathcal{A} = B(\mathcal{H})$. If ϕ is a surjective, additive mapping satisfying any of the $*$-identities* (B1), (B2), (B3), *then there exists a unitary operator $u \in B(\mathcal{H})$ and a scalar $\lambda \in \{1, -1\}$ such that either $\phi(x) = \lambda uxu^*$ for all $x \in \mathcal{A}$ or $\phi(x) = \lambda u(x^{T*})u^*$ for all $x \in \mathcal{A}$.*

As a corollary, we also obtain the complete description of all additive mappings satisfying any of the three two-variable $*$-identities.

Corollary 3. *Let $\mathcal{A}$ be a Schatten class of compact operators acting on the separable Hilbert space $\mathcal{H}$ or let $\mathcal{A} = B(\mathcal{H})$. If $\phi : \mathcal{A} \to \mathcal{A}$ is a surjective, additive mapping that satisfies one of the two-variable $*$-identities*

$$\phi(xy^*) = \phi(x)\phi(y)^*,$$
$$\phi(x^*y) = \phi(x)^*\phi(y),$$

then there exists a unitary operator $u \in \mathcal{B}(\mathcal{H})$ *such that either* $\phi(x) = uxu^*$ *for all* $x \in \mathcal{A}$ *or* $\phi(x) = u(x^{T*})u^*$ *for all* $x \in \mathcal{A}$*. If* ϕ *fulfills the remaining two-variable* $*$*-identity*

$$\phi(x^*y^*) = \phi(x)^*\phi(y)^*,$$

then there exists a unitary operator $u \in B(\mathcal{H})$ *and a third root of identity* λ*, such that either* $\phi(x) = \lambda uxu^*$ *for all* $x \in \mathcal{A}$ *or* $\phi(x) = \lambda u(x^{T*})u^*$ *for all* $x \in \mathcal{A}$*.*

Proof. We give the proof only in the case of the first identity. The remaining two can be treated in a similar way.

From $\phi(xy^*) = \phi(x)\phi(y)^*$ we obtain

$$\phi(xyz^*) = \phi(x)\phi(zy^*)^* = \phi(x)[\phi(z)\phi(y)^*]^* = \phi(x)\phi(y)\phi(z)^*$$

and hence ϕ satisfies (A3). According to Corollary 2, without serious loss of generality we can assume that there exists a unitary operator u and a complex number λ of modulus one such that $\phi(x) = \lambda uxu^*$. It is obvious that $\lambda = 1$. □

Remark 4. We suspect that every surjective, linear mapping defined on a Schatten class that satisfies any of the $*$-identities of order n is necessarily of the form $\phi(x) = uxv$ with some unitary operators u, v. Unfortunately, this is only a conjecture, the proof is missing.

Observe that if a $*$-identity is given, then not every mapping of the form $\phi(x) = uxv$ must satisfy it.

Example. The condition of surjectivity cannot be removed from our results and even cannot be replaced by other conditions like injectivity, for example. In order to see this, let $\mathcal{H}$ be infinite dimensional and let $u \in B(\mathcal{H})$ be such that $u^*u = 1$ and $uu^* \neq 1$, i.e., for instance, let u be a unilateral shift. Then $\phi(x) = ux$ satisfies (A2), ϕ is injective but not surjective.

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