Linear relations

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Introduction

Let X and Y be vector spaces over a field K. A relation S from X into Y (i.e., a set $S \subset X \times Y$ such that $S(x) = \{y \in Y\colon (x, y) \in S\} \neq \emptyset$ for all $x \in X$) is said to be linear if

$$S(x)+S(y) \subset S(x+y) \quad \text{and} \quad \lambda S(x) \subset S(\lambda x)$$

for all $x, y \in X$ and $\lambda \in K$, where the linear operations for subsets of Y are to be understood in the usual sense [2, 8].

This definition extends the notion of a linear function, and is mainly motivated by the fact that the inverse of a linear function is a linear relation.

Linear relations and linear set-valued mappings were probably first considered by R. ARENS [1] and C. BERGE [2], respectively. Some relevant results can also be found in [3—4, 9—11, 14—16].

In this paper, we initiate a systematic study of linear relations. In §1, the definition of linear relations is given, and several important examples are listed. The necessary verifications are lengthy enough, and are therefore omitted.

In § 2, some simple properties of linear relations are derived first, and then some basic operations for linear relations are introduced. It turns out that the family of all linear relations from X into Y forms an interesting algebraic structure.

In § 3, we prove that if f is a selection for a linear relation S (i.e., a choice function for the family of all relation classes $S(x)$), then $S(x)=f(x)+S(0)$ holds for all x. Furthermore, it is shown that this simple fact has several interesting consequences. For example, this implies that every reflexive linear relation is an equivalence relation.

In § 4, some special linear selections for linear relations are constructed. For example, we prove that for every linear relation S there exists a linear selection f such that $S(x)=S(y)$ implies $f(x)=f(y)$. Moreover, that there exists a linear function φ such that $S=\varphi^{-1}\circ f$.

Finally, we remark that in a continuation of this paper, quotient spaces defined by linear relations will be investigated, where much more topological considerations will be involved.

§ 1. Definition and examples

Definition 1.1. Let X and Y be vector spaces over a field (or a skew-field) K. A relation S from X into Y is said to be linear if

$$S(x)+S(y) \subset S(x+y) \quad \text{and} \quad \lambda S(x) \subset S(\lambda x)$$

for all $x, y \in X$ and $\lambda \in K$.

Linear relations from X into X are simply said to be linear relations on X.

Remark 1.2. Note that if S is a function, then the above definition reduces to the usual one.

Example 1.3. Let f be a linear function from X onto Y. Then f^{-1} is a linear relation from Y onto X.

Example 1.4. Let X be the vector space of all real- or complex-valued differentiable functions on an interval I in $\mathbf{R}$, and $Y=\{x' \colon x \in X\}$. Denote D the differential operator defined on X by $D(x)=x'$. Then the indefinite integral $\int = D^{-1}$ is a linear relation from Y onto X.

Example 1.5. Let X be an algebra over K, and let Y be a vector space over K and a right module over X such that $\lambda(yx)=(\lambda y)x=y(\lambda x)$ for all $\lambda \in K$, $y \in Y$ and $x \in X$. (In this case, we briefly say that Y is a right vector module over X.) Moreover, let f be a function from a subset D of X into Y such that $f(x)y=f(y)x$ for all $x, y \in D$. Then

$$\tilde{f} = \{(x, y) \in X \times Y \colon \; yz = f(z)x \text{ for all } z \in D\}$$

is a linear relation such that $f \subset \tilde{f}$. Moreover, if X is commutative, then the domain $\tilde{D}$ of $\tilde{f}$ is an ideal in X and $\tilde{f}(x)y \subset \tilde{f}(xy)$ for all $x \in \tilde{D}$ and $y \in X$.

Example 1.6. Let Γ be a directed set, and X be the vector space of all convergent nets in a topological vector space Y defined on Γ. Then the convergence

$$\lim_{\Gamma} = \{(x, y) \in X \times Y \colon \; x \to y\}$$

is a linear relation from X onto Y.

Example 1.7. Let X and Y be topological vector spaces such that Y is complete, and let f be a continuous linear function from a dense subspace M of X into Y. Then

$$\bar{f} = \{(x, y) \in X \times Y \colon \; y \in \lim_{t \to x} f(t)\}$$

is a linear relation from X into Y. (The above notation is consistent since $\bar{f}$ coincides with the closure of f in $X \times Y$.)

Example 1.8. Let Q be a measure space and Y be a topological vector space. Denote X the vector space of all functions x from Q into Y for which there exists $y \in Y$ such that $\lambda(y) = \int_Q \lambda \circ x$ for all $\lambda \in Y^*$, where Y^* is the dual space of Y. Then the weak integral

$$\int_Q = \left\{(x, y)\in X\times Y\colon \lambda(y) = \int_Q \lambda\circ x \text{ for alll } \lambda\in Y^*\right\}$$

is a linear relation from X into Y.

Example 1.9. Let Y be a Hilbert-space and T be a linear operator from a subspace M of Y into Y. Denote X the set of all $x\in Y$ for which the functional $z\to\langle T(z), x\rangle$ is continuous. Then the adjoint

$$T^* = \{(x, y)\in X\times Y\colon \langle T(z), x\rangle = \langle z, y\rangle \text{ for all } z\in M\}$$

of T is a linear relation from X into Y.

Example 1.10. Let f be a linear function from X into Y and M be a subspace of Y. Then the relation S from X into Y defined by

$$S(x) = f(x)+M$$

is a linear relation from X into Y. (We shall see later that every linear relation is of this form.)

Notes and comments. Definition 1.1 and Examples 1.3 and 1.9, apart from slight differences, are due to R. Arens [1]. Example 1.4 was given by C. Berge [2].
Topological vector spaces are always supposed to be defined over the field $K=\mathbf{R}$ or $\mathbf{C}$, and of course, they are not supposed to be Hausdorff.

§ 2. Properties and operations

Theorem 2.1. *Let S be a linear relation from X into Y. Then the indeterminancy $S(0)$ of S[10] is a subspace of Y. Moreover, S is a function if and only if $S(0)=\{0\}$.*

Proof. This follows at once from Definition 1.1 and from $S(x)-S(x)\subset S(0)$.

Theorem 2.2. *Let X and Y be vector spaces over K and S be a relation from X into Y. Then the following properties are pairwise equivalent:*

(i) *S is linear;*

(ii) *S is a subspace of $X\times Y$;*

(iii) *$S(x+y)=S(x)+S(y)$, $0\in S(0)$ and $S(\lambda x)=\lambda S(x)$ for all $x, y\in X$ and $0\neq\lambda\in K$.*

Proof. It is clear that (i) and (ii) are equivalent. To prove that (i) implies (iii), observe that $S(x+y)-S(y)\subset S(x)$ and $\lambda^{-1}S(\lambda x)\subset S(x)$, whence it follows that $S(x+y)\subset S(x)+S(y)$ and $S(\lambda x)\subset\lambda S(x)$.

Corollary 2.3. *Let S be a linear relation from X into Y. Then*

$$S(A+B) = S(A)+S(B) \quad \text{and} \quad S(\Lambda A) = \Lambda S(A)$$

for all $A, B\subset X$ and $0\notin\Lambda\subset K$.

PROOF. Since unions are preserved under relations, we have

$$S(A+B) = S\Big(\bigcup_{\substack{a\in A\\ b\in B}} \{a+b\}\Big) = \bigcup_{\substack{a\in A\\ b\in B}} S(a+b) = \bigcup_{\substack{a\in A\\ b\in B}} (S(a)+S(b)) = S(A)+S(B)$$

and

$$S(\Lambda A) = S\Big(\bigcup_{\substack{\lambda\in \Lambda\\ a\in A}} \{\lambda a\}\Big) = \bigcup_{\substack{\lambda\in \Lambda\\ a\in A}} S(\lambda a) = \bigcup_{\substack{\lambda\in \Lambda\\ a\in A}} \lambda S(a) = \Lambda S(A)$$

by (iii) in Theorem 2.2.

Theorem 2.4. *Let S and T be linear relations from X into Y and $\lambda\in K$. Then the relations $S+T$ and λS from X into Y defined by*

$$(S+T)(x) = S(x)+T(x) \quad \textit{and} \quad (\lambda S)(x) = \lambda S(x)$$

are linear relations from X into Y.

PROOF. Simple computation.

Remark 2.5. Observe that the family $\mathscr{S}(X, Y)$ of all linear relations from X into Y does not form a vector space. However, it is noteworthy that only two of the axioms of a vector space are not satisfied. (Note that $(\lambda+\mu)S=\lambda S+\mu S$ if $\lambda+\mu\neq 0$.)

Theorem 2.6. *Let S be a linear relation from X into Y and T be a linear relation from Y into Z. Then $T\circ S$ is a linear relation from X into Z.*

PROOF. A direct application of Corollary 2.3.

Theorem 2.7. *Let S be a linear relation from X onto Y. Then S^{-1} is a linear relation from Y onto X.*

PROOF. Quite obvious from (ii) in Theorem 2.2.

Theorem 2.8. *Let X and Y be vector spaces over K, E a basis for X, and $R\subset X\times Y$ such that $E\subset \operatorname{dom} R$. Then there exists a smallest linear relation S from X into Y such that $R\subset S$.*

PROOF. Let S be the intersection of all linear relations from X into Y containing R. Then, by (ii) in Theorem 2.2, it is clear that S is a linear relation. Thus, we have

$$\sum_{e\in E} \hat{x}(e)R(e) \subset \sum_{e\in E} \hat{x}(e)S(e) \subset S\Big(\sum_{e\in E} \hat{x}(e)e\Big) = S(x)$$

for each $x\in X$, where $\hat{x}$ denotes the unique function from E into K with finite support such that $x=\sum_{e\in E} \hat{x}(e)e$. This shows that X is the domain of S.

Corollary 2.9. *Let $\mathscr{S}=\mathscr{S}(X, Y)$ be the family of all linear relations from X into Y. Then $\mathscr{S}^*=\mathscr{S}\cup\{\emptyset\}$ is a complete set lattice.*

PROOF. With set inclusion $\mathscr{S}$ is a partially ordered set. Moreover, by Theorem 2.8., every nonempty subset of $\mathscr{S}$ has a least upper bound. Hence, the assertion is quite obvious. (One encounters a similar situation in connection with filters [18].)

Theorem 2.10. *Let X and Y be topological vector spaces and S be a linear relation from X into Y. Then the closure $\bar{S}$ of S in $X \times Y$ is a linear relation from X into Y.*

PROOF. This is quite obvious by (ii) in Theorem 2.2, since the closure of a linear subspace of a topological vector space is also a linear subspace.

Theorem 2.11. *Let S be a linear relation from X into a topological vector space Y. Then*

$$\{(x, y) \in X \times Y:\ y \in \overline{S(x)}\}$$

is a linear relation from X into Y.

PROOF. This follows immediately from the corresponding properties of the closure operation.

Remark 2.12. If S is as in Theorem 2.11 and $S(x)^{\circ} \neq \emptyset$ for some $x \in X$, then $S = X \times Y$.

Notes and comments. Some of the results of this paragraph are also due to R. ARENS [1] and C. BERGE [2].

The algebraic structure of certain general operation preserving relations has been intensively studied by D. PUPPE [11].

§ 3. Selections

Definition 3.1. A function f defined on the domain of a relation S is called a selection for S if $f \subset S$.

Remark 3.2. Note that f is a selection for a relation S if and only if f is a choice function for the family of all relation classes $S(x)$ of S. Note also that the statement that "each relation has a selection" is equivalent to the axiom of choice.

Theorem 3.3. *Let S be a linear relation from X into Y and f be a selection for S. Then*

$$S(x) = f(x) + S(0)$$

for all $x \in X$.

PROOF. Clearly $f(x) + S(0) \subset S(x) + S(0) = S(x)$. On the other hand, $S(x) - f(x) \subset S(x) - S(x) = S(0)$, and hence $S(x) \subset f(x) + S(0)$.

Corollary 3.4. *Let S be a linear relation from X into Y. Then $S(x)$ is a linear manifold (affine subspace) in Y* [8] *for all $x \in X$.*

PROOF. This follows immediately from Theorems 2.1 and 3.3.

Corollary 3.5. *Let S be a linear relation from X into Y. Then S is semi-single-valued* [2].

PROOF. By Theorem 3.3, it is clear that $S(x) \cap S(y) \neq \emptyset$ implies that $S(x) = S(y)$.

Corollary 3.6. *Let S be a linear relation from X into Y. Then $S(x)$ has the same cardinality for each $x \in X$.*

Proof. This is quite obvious from Theorem 3.3.

Corollary 3.7. *Let S be a reflexive linear relation on X. Then*

$$S(x) = x + S(0)$$

for all $x \in X$, and so S is an equivalence relation on X.

Proof. Since S is reflexive, the identity function of X is a selection for S. The remaining part is quite obvious by Theorems 3.3 and 2.1.

Remark 3.8. Observe that there is a one-to-one correspondence between linear equivalence relations on X and subspaces of X, respectively.

Observe also that the family of all linear equivalence relations on X forms an interesting algebraic structure with the operations treated in § 2. In particular, it is a complete set lattice.

Example 3.9. If f is a linear function from X onto X different from the identity function of X, then f^{-1} is a non-reflexive linear relation on X.

Corollary 3.10. *Let S be a linear relation from X into Y. Then $S^{-1} \circ S$ is a linear equivalence relation on X, and*

$$S^{-1} \circ S = S^{-1} \circ f = f^{-1} \circ S$$

for each selection f for S.

Proof. By Theorems 2.7 and 2.6 and Corollary 3.7, it is clear that $S^{-1} \circ S$ is a linear equivalence relation on X. If f is a selection for S, then by Theorem 3.3, $S^{-1}(f(x)) = x + S^{-1}(0)$ for all $x \in X$, and so $S^{-1} \circ f$ is independent of f. This guarantees that $S^{-1} \circ f = S^{-1} \circ S$. Finally, $f^{-1} \circ S = (S^{-1} \circ f)^{-1} = (S^{-1} \circ S)^{-1} = S^{-1} \circ S$.

Corollary 3.11. *If S is a reflexive linear relation, then*

$$S^{-1} \circ S = S.$$

Proof. This follows immediately from Corollary 3.10, since in this case the identity function is a selection for S.

Corollary 3.12.*) *If S is a linear relation, then*

$$S \circ S^{-1} \circ S = S.$$

Proof. By Corollaries 3.10 and 2.3, $(S \circ S^{-1} \circ S)(x) = S(x + S^{-1}(0)) = S(x) + S(S^{-1}(0)) = S(x) + S(0) = S(x)$ for all $x \in X$.

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*) Meantime, we observed that for an arbitrary relation S, we have $S \circ S^{-1} \circ S = S$ if and only if S is semi-singlevalued in the sense that $S(x) \cap S(y) \neq \emptyset$ implies that $S(x) = S(y)$.

Notes and comments. Corollary 3.7 for certain general operation preserving relations was proved by G. D. FINDLAY [3] in a direct way.

Corollary 3.12 for additive relations introduced by S. MCLANE is stated in Exercise 1 of [10].

The most striking result of this paragraph is Corollary 3.10.

§ 4. Linear selections

Theorem 4.1. Let S be a linear relation from X into Y, $0\neq\xi\in X$ and $\eta\in S(\xi)$. Then there exists a linear selection f for S such that $f(\xi)=\eta$.

PROOF. Let E be a basis for X such that $\xi\in E$, and for each $x\in X$, denote $\hat{x}$ the unique function from E into K with finite support such that $x=\sum_{e\in E}\hat{x}(e)e$. Choose a selection φ for $S|E$ such that $\varphi(\xi)=\eta$, and define the function f on X by

$$f(x)=\sum_{e\in E}\hat{x}(e)\varphi(e).$$

Then, it is clear that f is linear and $f(\xi)=\eta$. Moreover, we have

$$f(x)\in\sum_{e\in E}\hat{x}(e)S(e)\subset S\Big(\sum_{e\in E}\hat{x}(e)e\Big)=S(x)$$

for all $x\in X$.

Corollary 4.2. *Let S be a linear relation from X into Y and $\mathscr{F}$ be the family of all linear selections for S. Then $S=(\cup\mathscr{F})\cup(\{0\}\times S(0))$.*

PROOF. This follows at once from Theorem 4.1.

Corollary 4.3. *Every linear relation S can be written in the form*

$$S=f+C \quad \text{or} \quad S=R\circ f$$

where f is a linear function, C is a constant linear relation and R is a linear equivalence relation.

PROOF. This follows immediately from Theorems 4.1 and 3.3.

Corollary 4.4. *Let X and Y be nontrivial vector spaces over a field K of characteristic 0, and denote $\mathscr{S}(X,Y)$ the family of all linear relations from X into Y. Then*

$$\operatorname{card}\mathscr{S}(X,Y)=\max\{(\operatorname{card}Y)^{\dim X},2^{\dim Y}\}.$$

PROOF. The cardinal number of the family of all linear functions from X into Y is $p=(\operatorname{card}Y)^{\dim X}$. The cardinal number of the family of all subspaces of Y is $q=2^{\dim Y}$. Thus, we have $\max\{p,q\}\leqq\operatorname{card}\mathscr{S}(X,Y)$. Moreover, by Corollary 4.3, it is clear that $\operatorname{card}\mathscr{S}(X,Y)\leqq pq$. Finally, since by the assumptions p is infinite, $pq=\max\{p,q\}$. (It is also known that $\operatorname{card}Y=(\operatorname{card}K)^{\dim Y}$ if $\dim Y$ is finite, and $\operatorname{card}Y=\max\{\operatorname{card}K,\dim Y\}$ if $\dim Y$ is infinite [7, p. 32].)

Theorem 4.5. *Let S be a linear relation from X into Y. Then there exists a linear selection f for S such that $S(x)=S(y)$ implies that $f(x)=f(y)$.*

PROOF. By Theorems 2.7 and 2.1, the kernel $S^{-1}(0)$ of S is a subspace of X. Let E be a basis for X such that $E\cap S^{-1}(0)$ is a basis for $S^{-1}(0)$ if $S^{-1}(0)\neq\{0\}$, and for each $x\in X$, let $\hat{x}$ be as in the proof of Theorem 4.1. Since $0\in S(x)$ for each $x\in S^{-1}(0)$, there exists a selection φ for $S|E$ such that $\varphi(e)=0$ for all $e\in E\cap S^{-1}(0)$. Define the function f on X by the same formula as in the proof of Theorem 4.1. Then f is a linear selection for S such that $f(x)=0$ for all $x\in S^{-1}(0)$. Thus, if $S(x)=S(y)$, then $f(x)+S(0)=f(y)+S(0)$, and hence $x-y\in f^{-1}(S(0))=S^{-1}(f(0))=S^{-1}(0)$ by Corollary 3.10. Consequently $f(x-y)=0$, i.e., $f(x)=f(y)$.

Corollary 4.6. *Every linear relation S can be written in the form*

$$S=\varphi^{-1}\circ f,$$

where f is a linear selection for S and φ is a linear function.

PROOF. Let S and f be as in Theorem 4.5. Then, we may define a function φ on the range $S(X)$ of S by $\varphi(y)=f(x)$, if $y\in S(x)$. Then, it is clear that φ is a linear function from $S(X)$ onto $f(X)$, and moreover, we have $S(x)=\varphi^{-1}(f(x))$ for all $x\in X$.

Corollary 4.7. *For every linear relation S, the linear equivalence relation $S^{-1}\circ S$ can be written in the form*

$$S^{-1}\circ S=f^{-1}\circ f,$$

where f is a linear selection for S.

PROOF. If S is a linear relation, and f, φ are as in Corollary 4.6, then $S^{-1}\circ S=f^{-1}\circ\varphi\circ\varphi^{-1}\circ f=f^{-1}\circ f$.

Corollary 4.8. *Every reflexive linear relation S can be written in the form*

$$S=f^{-1}\circ f,$$

where f is a linear selection for S.

PROOF. This follows immediately from Corollaries 4.7 and 3.11.

Theorem 4.9. *Let S be a linear relation from X into Y, and let ψ be a linear function from a subspace M of X into Y such that $\psi\subset S$. Then there exists a linear selection f for S such that $\psi\subset f$.*

PROOF. Let E be a basis for X such that $E\cap M$ is a basis for M, if $M\neq\{0\}$; and for each $x\in X$, let $\hat{x}$ be as in the proof of Theorem 4.1. Choose a selection φ for $S|E$ such that $\varphi(e)=\psi(e)$ for all $e\in E\cap M$, and define the function f on X as in the proof of Theorem 4.1. Then, it is clear that f has the required properties.

Corollary 4.10. *Let X and Y be topological vector spaces such that Y is complete, and let f be a continuous linear mapping of a dense subspace M of X into Y. Then there exists a continuous linear mapping F from X into Y such that $f\subset F$.*

PROOF. Let $\bar{f}$ be as in Example 1.7. Then, by Theorem 4.9, there exists a linear selection F for $\bar{f}$ such that $f \subset F$. Thus, we have

$$F(x) \in \lim_{\substack{t \to x \\ t \in M}} F(t)$$

for all $x \in X$, and this guarantees that F is continuous.

Notes and comments. A weaker form of Theorem 4.1 was proved in [14] by the authors.

Some of the results of this paragraph can also be extended to the case of free modules over a ring with identity.

Corollary 4.10 for the case when Y is Hausdorff can be found in most treatises on topological vector spaces [8].

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