A theorem on the product of distributions

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In the following ϱ denotes a fixed infinitely differentiable function having the following properties:

(1) $\varrho(x) = 0 \quad \text{for} \quad |x| \geqq 1,$

(2) $\varrho(x) \geqq 0,$

(3) $\varrho(x) = \varrho(-x),$

(4) $\int_{-1}^{1} \varrho(x)\,dx = 1.$

The function δ_n is now defined by

$$\delta_n(x) = n\varrho(nx),$$

for $n=1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function δ.

If now f is an arbitrary distribution, we define the function f_n by

$$f_n(x) = f * \delta_n = \int_{-1/n}^{1/n} f(x-t)\,\delta_n(t)\,dt,$$

for $n=1, 2, \ldots$. It follows that $\{f_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution f.

In [1], the product of two distributions f and g was defined as the limit of the sequence $\{f_n g_n\}$, provided that this sequence was regular so that the limit did in fact exist.

With this definition of the product of two distributions we have

Theorem. *Suppose that f and g are even and odd distributions respectively and that they are ordinary continuous functions on every open interval not containing the origin. Suppose further that there exist constants α, β, c and k with $\alpha+\beta>-2$ and $c, k>0$ such that*

$$|f(x)| < k\,|x|^{\alpha}, \quad |g(x)| < k\,|x|^{\beta}, \quad \text{for} \quad 0 < |x| < c$$

and

$$|f_n(x)| < kn^{-\alpha}, \quad |g_n(x)| < kn^{-\beta}, \quad \text{for} \quad |x| \leqq 2/n \quad \textit{and} \quad n > 2/c.$$

Then the sequence $\{f_n g_n\}$ is regular and so the product fg exists.

PROOF. We will choose c_1 such that $0<c_1<c$. Then if $2/n \leqq x \leqq c_1 - 1/n$, we have

$$\begin{aligned}\left|(|x|^\alpha)_n\right| &= |x|^\alpha * \delta_n(x)\\ &= \int_{-1/n}^{1/n} (x-t)^\alpha \delta_n(t)\,dt\\ &\leqq \sup_t \{|\delta_n(t)|\} \int_{-1/n}^{1/n} (x-t)^\alpha\,dt\\ &= n \sup_t \{\varrho(t)\}\, 2(x-a)^\alpha/n\end{aligned}$$

on using the mean-value theorem for integrals, where $-1/n \leqq a \leqq 1/n$. It follows that

$$|(|x|^\alpha)_n| \leqq 2hc_1^\alpha, \quad \text{for} \quad 2/n \leqq x \leqq c_1 - 1/n$$

where

$$h = \sup_t \{\varrho(t)\}.$$

By the symmetry, it now follows that

$$\left|(|x|^\alpha)_n\right| \leqq 2hc_1^\alpha, \quad \text{for} \quad 2/n \leqq |x| \leqq c_1 - 1/n.$$

We therefore have the results

$$|f_n(x)\, g_n(x)| < k^2 n^{-\alpha-\beta}, \quad \text{for} \quad |x| \leqq 2/n \quad \text{and} \quad n > 2/c$$

and

$$|f_n(x)\, g_n(x)| < 4h^2 k^2 c_1^{\alpha+\beta}, \quad \text{for} \quad 2/n \leqq |x| \leqq c_1 - 1/n.$$

We now note that since f is an even distribution and g is an odd distribution, $f_n g_n$ must be an odd function. Thus if φ is an arbitrary infinitely differentiable test function with compact support, we have

$$(f_n\, g_n,\, \varphi) = \int_0^\infty f_n(x) g_n(x) \{\varphi(x) - \varphi(-x)\}\, dx$$

and so

$$\begin{aligned}&\left|(f_n\, g_n,\, \varphi) - \int_{c_1}^\infty f_n(x) g_n(x) \{\varphi(x) - \varphi(-x)\}\, dx\right|\\ &= \left|\int_0^{c_1} f_n(x) g_n(x) \{\varphi(x) - \varphi(-x)\}\, dx\right|\\ &\leqq \int_0^{2/n} |f_n(x) g_n(x)|\, |\varphi(x) - \varphi(-x)|\, dx + \int_{2/n}^{c_1} |f_n(x) g_n(x)|\, |\varphi(x) - \varphi(-x)|\, dx\\ &\leqq k^2 n^{-\alpha-\beta} \int_0^{2/n} |\varphi(x) - \varphi(-x)|\, dx + 4h^2 k^2 c_1^{\alpha+\beta} \int_{2/n}^{c_1} |\varphi(x) - \varphi(-x)|\, dx\end{aligned}$$

for $n \geqq 3/c_1$.

Using the mean value theorem we have

$$|\varphi(x)-\varphi(-x)| \leqq b\,|x|$$

for $|x| \leqq c$, where

$$b = \sup\{|\varphi'(x)+\varphi'(-x)|:\ |x| \leqq c\}.$$

It follows that for arbitrary $\varepsilon > 0$,

$$4h^2k^2c_1^{\alpha+\beta}\int_{2/n}^{c_1}|\varphi(x)-\varphi(-x)|\,dx \leqq 4bh^2k^2c_1^{\alpha+\beta}\int_{2/n}^{c_1}x\,dx =$$
$$= 2bh^2k^2c_1^{\alpha+\beta}(c_1^2-4/n^2) \leqq 2bh^2k^2c_1^{2+\alpha+\beta} < \varepsilon$$

for all c_1 with $0 < c_1 \leqq \eta$, for some fixed $\eta > 0$, since $2+\alpha+\beta > 0$. Further

$$k^2n^{-\alpha-\beta}\int_0^{2/n}|\varphi(x)-\varphi(-x)|\,dx \leqq bk^2n^{-\alpha-\beta}\int_0^{2/n}x\,dx = 2bk^2n^{-2-\alpha-\beta} < \varepsilon$$

for all n greater than some N. Thus

$$\left|(f_ng_n, \varphi) - \int_{c_1}^{\infty} f_n(x)g_n(x)\{\varphi(x)-\varphi(-x)\}\,dx\right| < 2\varepsilon$$

if $n > N$ and $c_1 \leqq \eta$.

Using the fact that f and g are continuous functions for $x \geqq c_1$, the sequence $\{f_n(x)g_n(x)\}$ converges to $f(x)g(x)$ and so

$$\lim_{n\to\infty}\int_{c_1}^{\infty} f_n(x)g_n(x)\{\varphi(x)-\varphi(-x)\}\,dx = \int_{c_1}^{\infty} f(x)g(x)\{\varphi(x)-\varphi(-x)\}\,dx.$$

Since ε is arbitrary, it now follows that the sequence $\{(f_ng_n, \varphi)\}$ is convergent for all test functions φ and so the sequence $\{f_ng_n\}$ is regular. This proves the existence of the product fg.

Having shown that the sequence $\{(f_ng_n, \varphi)\}$ is convergent we have the extra result that

$$\lim_{n\to\infty}\left|(f_ng_n, \varphi) - \int_{c_1}^{\infty} f_n(x)g_n(x)\{\varphi(x)-\varphi(-x)\}\,dx\right| =$$
$$= \left|(fg, \varphi) - \int_{c_1}^{\infty} f(x)g(x)\{\varphi(x)-\varphi(-x)\}\,dx\right| \leqq 2\varepsilon$$

for all $c_1 \leqq \eta$, η depending on ε. Since ε is arbitrary it follows that

$$(1) \qquad (fg, \varphi) = \lim_{\eta\to 0}\int_{\eta}^{\infty} f(x)g(x)\{\varphi(x)-\varphi(-x)\,dx.$$

The conditions imposed on f and g in the theorem are stronger than necessary. It is in fact easy to see than the conditions imposed on f and g need only hold on the open interval $(-c, c)$. Outside of this interval we only need to know that the product fg exists. Equation (1) would then of course not necessarily hold. The theorem can of course also be generalized by translation.

We will now consider particular products of distributions.

The distribution $\delta^{(2r)}$ is an even distribution for $r=0, 1, 2, \ldots$ and $\delta^{(2r)}(x)=0$ for $|x|>0$. Further

$$\delta_n^{(2r)}(x) = n^{2r+1}\varrho^{(2r)}(nx)$$

and so

$$|\delta_n^{(2r)}(x)| \leqq \begin{cases} \sup\limits_x \{|\varrho^{(2r)}(x)|\} n^{2r+1}, & \text{for} \quad |x| < 1/n \\ 0, & \text{for} \quad |x| \geqq 1/n. \end{cases}$$

The distribution $\delta^{(2r-1)}$ is an odd distribution for $r=1, 2, \ldots$ and $\delta^{(2r-1)}(x)=0$ for $|x|>0$. Further

$$|\delta_n^{(2r-1)}(x)| \leqq \begin{cases} \sup\limits_x \{|\varrho^{(2r-1)}(x)|\} n^{2r}, & \text{for} \quad |x| < 1/n \\ 0, & \text{for} \quad |x| \geqq 1/n. \end{cases}$$

The distribution $|x|^\alpha$ is an ordinary summable even function when $\alpha>-1$ and in this case we have

$$\begin{aligned} |(|x^\alpha|)_n| &= \int_{-1/n}^{1/n} |x-t|^\alpha \delta_n(t)\, dt \\ &\leqq \sup_t \{\delta_n(t)\} \int_{-1/n}^{1/n} |x-t|^\alpha\, dt \\ &\leqq n \sup_t \{\varrho(t)\}(\alpha+1)^{-1}\{|x-1/n|^{\alpha+1}+|x+1/n|^{\alpha+1}\} \\ &\leqq 2\cdot 3^{\alpha+1}(\alpha+1)^{-1} \sup_t \{\varrho(t)\}\, n^{-\alpha} \end{aligned}$$

for $|x| \leqq 2/n$.

Similarly, the distribution $\operatorname{sgn} x|x|^\alpha$ is an ordinary summable odd function when $\alpha>-1$ and in this case we have

$$\begin{aligned} \left|(\operatorname{sgn} x|x|^\alpha)_n\right| &= \Big|\int_{-1/n}^{1/n} \operatorname{sgn}(x-t)|x-t|^\alpha \delta_n(t)\, dt\Big| \\ &\leqq \int_{-1/n}^{1/n} |x-t|^\alpha \delta_n(t)\, dt \\ &\leqq 2\cdot 3^{\alpha+1}(\alpha+1)^{-1} \sup_t \{\varrho(t)\}\, n^{-\alpha} \end{aligned}$$

for $|x| \leqq 2/n$. We also obviously have

$$\left|\operatorname{sgn} x|x|^\alpha\right| = |x|^\alpha$$

for $|x|>0$ and all values of α.

Now suppose that either $-2r-1<\alpha<-2r$ or $-2r<\alpha<-2r+1$, for some positive integer r. Then $|x|^\alpha$ is an even distribution, $|x|^{\alpha+2r}$ is an ordinary summable function and

$$|x|^\alpha = \prod_{i=1}^{2r} (\alpha+i)^{-1} \frac{d^{2r}}{dx^{2r}} |x|^{\alpha+2r}.$$

It follows that

$$\left|(|x|^{\alpha})_n\right| = \prod_{i=1}^{2r} |\alpha+i|^{-1} \Big| \int_{-1/n}^{1/n} |x-t|^{\alpha+2r} \delta_n^{(2r)}(t)\, dt \Big|$$

$$\leqq \prod_{i=1}^{2r} |\alpha+i|^{-1} \sup_t \{|\delta_n^{(2r)}(t)|\} \int_{-1/n}^{1/n} |x-t|^{\alpha+2r}\, dt$$

$$\leqq \prod_{i=1}^{2r+1} |\alpha+i|^{-1} n^{2r+1} \sup_t \{|\varrho^{(2r)}(t)|\} \{|x-1/n|^{\alpha+2r+1} + |x+1/n|^{\alpha+2r+1}\}$$

$$\leqq 2\cdot 3^{\alpha+2r+1} \prod_{i=1}^{2r+1} |\alpha+i|^{-1} \sup_t \{|\varrho^{(2r)}(t)|\} n^{-\alpha}$$

for $|x| \leqq 2/n$.

Similarly, if either $-2r-1 < \alpha < -2r$ or $-2r < \alpha < -2r+1$, then $\operatorname{sgn} x|x|^{\alpha}$ is an odd distribution and

$$\left|(\operatorname{sgn} x|x|^{\alpha})_n\right| \leqq 2\cdot 3^{\alpha+2r+1} \prod_{i=1}^{2r+1} |\alpha+i|^{-1} \sup_t \{|\varrho^{(2r)}(t)|\} n^{-\alpha}$$

for $|x| \leqq 2/n$.

When α takes the value of a negative integer $-r$, we have $|x|^{-r}$ is an even distribution, $\ln|x|$ and $\operatorname{sgn} x \ln|x|$ are ordinary summable functions and

$$(r-1)!\,|x|^{-r} = \begin{cases} -\dfrac{d^r}{dx^r} \ln|x|, & \text{for } r \text{ even} \\[2mm] \dfrac{d^r}{dx^r}(\operatorname{sgn} x \ln|x|), & \text{for } r \text{ odd.} \end{cases}$$

It follows that

$$(r-1)!\,(|x|^{-r})_n \leqq \int_{-1/n}^{1/n} \left|\ln|x-t|\,\delta_n^{(r)}(t)\right| dt \leqq$$

$$\leqq n^{r+1} \sup_t \{|\varrho^{(r)}(t)|\} \int_{-1/n}^{1/n} \left|\ln|x-t|\right| dt \leqq$$

$$\leqq n^{r+1} \sup_t \{|\varrho^{(r)}(t)|\} \{(x-1/n)\ln|x-1/n| - (x+1/n)\ln|x+1/n| + 2/n\} \leqq$$

$$\leqq \sup_t \{|\varrho^{(r)}(t)|\}(6\ln 3 + 6\ln n + 2) n^r \leqq$$

$$\leqq \sup_t \{|\varrho^{(r)}(t)|\}(6\ln 3 + 8) n^r \ln n \leqq 15 \sup_t \{|\varrho^{(r)}(t)|\} n^{r+\varepsilon}$$

for $|x| \leqq 2/n$ and arbitrary $\varepsilon > 0$, provided n is larger than some N which of course depends on ε. The distribution $|x|^{-r}$ can therefore be made to satisfy the conditions of the theorem by choosing c small enough.

Similarly, $\operatorname{sgn} x|x|^{-r}$ is an odd distribution and

$$(r-1)!\,\left|\operatorname{sgn} x|x|^{-r})_n\right| \leqq 15 \sup_t \{|\varrho^{(r)}(t)|\} n^{r+\varepsilon}$$

for $|x| \leqq 2/n$ and arbitrary $\varepsilon > 0$, provided $n > N$.

We will now consider the product of the distributions $|x|^\alpha$ and $\delta^{(2r+1)}$. It is easily seen that k can be found so that these two distributions satisfy the conditions of the theorem for $r=0, 1, 2, \ldots,$ provided that $\alpha>2r$. The product $|x|^\alpha \delta^{(2r+1)}$ therefore exists provided $\alpha>2r$.

Further, we know that for an arbitrary test function φ

$$(|x|^\alpha \delta^{(2r+1)}, \varphi) = \lim_{\eta\to 0} \int_\eta^\infty |x|^\alpha \delta^{(2r+1)}(x)\{\varphi(x)-\varphi(-x)\}\, dx = 0,$$

since

$$|x|^\alpha \delta^{(2r+1)}(x) = 0$$

for $x>0$. It follows that

$$|x|^\alpha \delta^{(2r+1)} = 0$$

for $\alpha>2r$ and $r=0, 1, 2, \ldots.$ This result was given in [2].

Similarly, it is easily seen that the two distributions $\operatorname{sgn} x\,|x|^\alpha$ and $\delta^{(2r)}$ satisfy the conditions of the theorem for $r=0, 1, 2, \ldots,$ provided that $\alpha>2r-1$. It is easily proved that we then have

$$\operatorname{sgn} x\,|x|^\alpha \delta^{(2r)} = 0.$$

This result was also given in [2].

Again it is easily seen that k can be found so that the two distributions $|x|^\alpha$ and $\operatorname{sgn} x\,|x|^\beta$ satisfy the conditions of the theorem provided that $\alpha+\beta>-2$. It then follows that for an arbitrary test function φ

$$\big(|x|^\alpha(\operatorname{sgn} x\,|x|^\beta), \varphi\big) = \lim_{\eta\to 0} \int_\eta^\infty x^{\alpha+\beta}\{\varphi(x)-\varphi(-x)\}\, dx =$$

$$= \int_0^\infty x^{\alpha+\beta}\{\varphi(x)-\varphi(-x)\}\, dx = (\operatorname{sgn} x\,|x|^{\alpha+\beta}, \varphi)$$

and so

$$|x|^\alpha(\operatorname{sgn} x\,|x|^\beta) = \operatorname{sgn} x\,|x|^{\alpha+\beta},$$

for $\alpha+\beta>-2$.

We finally note that the above results can in fact be generalized. It can be shown that $k>0$ exists such that

$$\big|(|x|^\alpha \ln^p |x|)_n\big|,\ \big|\operatorname{sgn} x\,|x|^\alpha \ln^p |x|_n\big| \leqq kn^{-\alpha}\ln^p n < kn^{-\alpha+\varepsilon}$$

for all $\varepsilon>0$ and sufficiently large n, for $p=1, 2, \ldots$ and

$$\big|(|x|^\alpha \ln^p |x|)_n\big|,\ \big|\operatorname{sgn} x\,|x|^\alpha \ln^p |x|)_n\big| \leqq kc^\alpha(-\ln c)^p < kc^{\alpha-\varepsilon}$$

for $0<c<1$, $2/n \leqq |x| \leqq c-1/n$, $\varepsilon>0$ and sufficiently large n, for $p=1, 2, \ldots.$

By choosing ε sufficiently small it follows that we have

$$(|x|^\alpha \ln^p |x|)\,\delta^{(2r+1)} = 0$$

for $\alpha>2r$ and $p, r=0, 1, 2, \ldots,$

$$(\operatorname{sgn} x\,|x|^\alpha \ln^p |x|)\,\delta^{(2r)} = 0$$

for $\alpha > 2r-1$ and $p, r = 0, 1, 2, \ldots$ and

$$(|x|^{\alpha} \ln^{p} |x|)(\operatorname{sgn} x |x|^{\beta} \ln^{q} |x|) = \operatorname{sgn} x |x|^{\alpha+\beta} \ln^{p+q} |x|$$

for $\alpha + \beta > -2$ and $p, q = 0, 1, 2, \ldots$.

References

[1] B. Fisher, The product of distributions, *Quart. J. Math. Oxford Ser. 2*, **22**, (1971), 291—298.
[2] B. Fisher, Some results on the product of distributions, *Proc. Camb. Phil. Soc.* **73**, (1973), 317—325.

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