A remark on universal spaces

By Á. P. Bosznay (Budapest)

Let us introduce the following well-known

Definition. The real separable $(X, \|\cdot\|)$ Banach space is said to be a universal space, iff for arbitrary real separable $(Y, |\cdot|)$ Banach space there exists a subspace of $(X, \|\cdot\|)$, isometrically isomorphic to $(Y, |\cdot|)$.

BANACH and MAZUR proved that $C[0, 1]$ is universal. This is a special case of the following [1. p. 229.]

Theorem. *Let Ω be a compact metric space. Then the assertions* A) *and* B) *are equivalent.*

A) $C(\Omega)$ *is a universal space,*

B) Ω *is an uncountable set.*

In this paper we construct some more universal spaces.

Theorem 1. *Let $A \subset C[0, 1]$ be a closed subalgebra and let $\dim A > 1$ (A is considered as a vector space). Then A is universal (with the* $C[0, 1]$ norm*).*

PROOF. Because of $\dim A > 1$, there exists a $h \in A, h(x_0) = \|h\|$, such that h is not a constant. Set

$$\bar{h} = \frac{h}{\|h\|} - \frac{h^2}{\|h^2\|} \in A.$$

The function $\bar{h}$ is not a constant, and $\bar{h}(x_0) = 0$.

It is clear that

$$[0, 1] = \cup \bar{h}^{-1}(c), \quad c \in [-\|\bar{h}\|, \|\bar{h}\|]$$

and this is a disjoint union. Let us denote $[y] = \bar{h}^{-1}(\bar{h}(y))$, where $y \in [0, 1]$ arbitrary.

Let us introduce the metric

$$d([x], [y]) = |\bar{h}(x) - \bar{h}(y)|$$

on the set

$$K = \{[x];\, x \in [0, 1]\}.$$

It is obvious that (K, d) is a compact metric space. Set

$$K' = \left\{[x];\, [x] \in K, \bar{h}(x) \geqq \frac{\|\bar{h}\|}{2}\right\}.$$

Clearly K' is uncountable, so by the above mentioned theorem, $C(K', d)$ is universal.

It suffices to prove that there exists a subspace in A, isometrical to $C(K', d)$.

Let $\varphi\colon K \to [0, 1]$ be continuous, such that

$$\varphi([x_0]) = 0, \quad \varphi([x]) = 1, \quad \text{if} \quad [x] \in K' \tag{1}$$

Let be $D\colon C(K', d) \to C(K, d)$ a linear isometrical extension (such a D exists, see [2] p. 365 Theorem 21.1.4).

Let be $T\colon C(K', d) \to C(K, d)$,

$$(Tg)([x]) = \varphi([x]) \cdot (Dg)([x]).$$

Clearly T is a linear isometrical operator.

From (1) we have $(Tg)([x_0]) = 0$, thus the Weierstrass-Stone theorem implies that for arbitrary $\varepsilon > 0$ a polynomial P_ε of $\bar{h}$ (with constant coefficient equals to 0) exists such that

$$\max_{[x] \in K} \left| P_\varepsilon(\bar{h}(x)) - (Tg)([x]) \right| < \varepsilon.$$

Since A is closed, $(Tg) \circ \bar{h} \in A$.

It is evident that the following $U\colon C(K, d) \to C[0, 1]$

$$(Uf)(t) = (f \circ \bar{h})(t)$$

operator is a linear isometric one, so

$$U \circ T\colon C[K', d] \to A$$

is a linear isometric one too. Qu.e.d.

Theorem 2. *Let $(X, \|\cdot\|)$ be a universal Banach space and let $(M, \|\cdot\|)$ be a finite codimensional closed subspace of it. Then $(M, \|\cdot\|)$ is also universal.*

Proof. A simple argument shows that it is enough to prove the following assertion.

Every one codimensional closed subspace of $C[0, 1]$ is universal.

If M is a one-codimensional closed subspace of $C[0, 1]$ then there exists a mapping $m\colon [0, 1] \to R$, such that $0 < \operatorname{tot\,var}_{[0,1]} m < +\infty$, and

$$g \in M \Leftrightarrow M(g) \equiv \int_0^1 g(x)\, dm(x) = 0. \tag{2}$$

Let be $0 \leqq a < a' < b' < b \leqq 1$, and let be m continuous at the points a and b. Further let be a, b satisfy

$$\operatorname{tot\,var}_{[a,b]} m \leqq \frac{1}{3} \operatorname{tot\,var}_{[0,1]} m = \frac{1}{3} \|M\|. \tag{3}$$

Hence there exists $s \in C[0, 1]$ such that $\|s\| \leqq 1$, $s(x) = 0$ for all $x \in [a, b]$ and

$$M(s) \geqq \frac{1}{3} \operatorname{tot\,var}_{[0,1]} m.$$

Let $g\in C[a', b']$ be arbitrary, and let us define $\bar{g}$: $[0, 1]\to R$ by

$$\bar{g}(x) = \begin{cases} 0, & \text{if} \quad x\in[0, a) \\ \dfrac{x-a}{a'-a}\, g(a'), & \text{if} \quad x\in[a, a') \\ g(x), & \text{if} \quad x\in[a', b'] \\ \dfrac{b-x}{b-b'}\, g(b'), & \text{if} \quad x\in(b', b] \\ 0, & \text{if} \quad x\in(b, 1]. \end{cases}$$

Using (3), we have

$$|M(\bar{g})| \leqq \frac{\|M\|}{3}\, \|\bar{g}\|.$$

Let $\hat{g}\in C[0, 1]$ be defined as

$$\hat{g}(x) = \bar{g}(x) - \frac{M(\bar{g})}{M(s)}\, s(x).$$

From (2) we obtain that $\hat{g}\in M$.

At the same time, the mapping $g\to\hat{g}$ is linear and isometrical, so M contains isometrically $C[a', b']$.

References

[1] R. B. HOLMES, Geometric functional analysis and its applications. Graduate texts in mathematics, vol. **24**. *New York—Heidelberg—Berlin* 1975.

[2] Z. SEMADENI, Banach spaces of continuous functions, vol. I. Monografie Matematyczne, Tom. **55**. *Warszawa* 1971.

Department of Probability Theory
L. Eötvös University, Budapest

(Received June 24, 1977.)