Inequalities for linear combinations of binomial moments

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Introduction

Let $A_1, A_2, \dots, A_n$ be a sequence of events on a given probability space. Put $S_{0,n}=1$ and

$$S_{k,n}=\sum P(A_{i_1}A_{i_2}\dots A_{i_k}), \quad k\geqq 1,$$

where the summation is over all subscripts satisfying $1\leqq i_1<i_2<\dots<i_k\leqq n$. It is easily seen that $S_{k,n}$ is the kth binomial moment of the number m_n of the A_j, $1\leqq j\leqq n$, which occur. For $r\geqq 0$, let $B_{r,n}=\{m_n=r\}$. Our aim is to investigate inequalities of the type

$$aP(B_{r,n})+\sum_{k=0}^{n} c_k S_{k,n}\geqq 0, \tag{1}$$

where a and c_k, $0\leqq k\leqq n$, are given constants. We permit the value zero for all coefficients. Therefore, (1) may reduce to an inequality among the binomial moments. On the other hand, if $a\neq 0$, (1) is an upper or lower bound on $P(B_{r,n})$ according as $a<0$ or $a>0$. Evidently, we can assume that a takes one of the values 0, 1 or -1. The coefficients c_k are, however, arbitrary and they may depend on r or n or both. Our investigation is an extension of a recent result by one of us in [4], where the dependence of c_k on n was implicitly assumed to be of a specific form. Here, we get rid of such restrictions. Hence, both the method of proof and the actual inequalities presented are of interest.

The method of polynomials

We present a method of proof of (1), which we term as the method of polynomials. It is related to one of the methods of GALAMBOS [4], in whose work it is implicitly assumed that the coefficients c_k either do not depend on n or if some of them do then they are of the same sign and they are monotonic in n. In addition, GALAMBOS [4] did not consider the case $a=0$ of (1), in which case it reduces to a linear inequality among binomial moments. Our method of proof is given in the following theorem, which does not make any precondition on the functions $c_k=c_k(n)$.

Theorem 1. *In* (1), *let the coefficients* $c_k=c_k(n)$ *be given real numbers which may depend on n. Then* (1) *holds on an arbitrary probability space for an arbitrary sequence* $A_1, A_2, \dots, A_n$ *of events if, and only if, it holds in the following special cases:*

$A_1, A_2, \ldots, A_n$ are independent with $P(A_j)=p$ for all j, and $c_k=c_k(N)$, where N runs through all integers greater than, or equal to n.

We shall call the sufficiency part of Theorem 1 the method of polynomials. Namely, by Theorem 1, (1) reduces to a set of polynomial inequalities.

We separate a part of the proof as a lemma. The lemma itself is interesting in that it shows that one does have to consider the inequalities obtained from (1) by replacing $c_k(n)$ by $c_k(N)$ for all $N \geqq n$.

Lemma 1. *If (1) holds on an arbitrary probability space for an arbitrary sequence A_j, $1 \leqq j \leqq n$, of events when $c_k=c_k(n)$ then it remains to hold with $c_k=c_k(N)$, where $N \geqq n$ is an arbitrary integer.*

Proof. Let $N>n$. Let $A_1, A_2, \ldots, A_n$ be arbitrary events and define A_j to be the empty set for $n<j\leqq N$. Since (1) holds for an arbitrary sequence of events, we obtain

$$aP(B_{r,N})+\sum_{k=0}^{N} c_k(N)\,S_{k,N} \geqq 0.$$

But $S_{k,N}=S_{k,n}$ and $P(B_{r,N})=P(B_{r,N})$ for our choice of the events A_j, $1\leqq j\leqq N$. Hence, the above inequality reduces to (1) with $c_k=c_k(N)$, what was to be proved.

Proof of Theorem 1. By an appeal to Lemma 1, the necessity of Theorem 1 is obvious. Therefore, only the sufficiency part needs proof. We shall follow the method of proof of Theorem 3 of [4]. Let $I(E)$ be the indicator of the event E. Let $J_{0,n}=1$ and

$$J_{k,n}=\sum I(A_{i_1})I(A_{i_2})\ldots I(A_{i_k}), \quad k\geqq 1,$$

where the summation is defined as for $S_{k,n}$. We first observe that (1) is equivalent to

$$aI(B_{r,n})+\sum_{k=0}^{n} c_k(n)\,J_{k,n} \geqq 0. \tag{2}$$

As a matter of fact, if we take expectation in (2), we get (1). On the other hand, if each A_i is either the empty set or the sure event then (1) reduces to (2). Hence, it suffices to show that the conditions of Theorem 1 imply the validity of (2). Let us thus assume that, in the stated special cases, (1) holds. This means that, for all $0\leqq p\leqq 1$ and for all integers $N\geqq n$,

$$a\binom{n}{r}p^r(1-p)^{n-r}+\sum_{k=0}^{n} c_k(N)\binom{n}{k}p^k \geqq 0. \tag{3}$$

By the choices $p=0$ and $p=1$, (3) yields

$$a\delta_{r,0}+c_0(N)\geqq 0, \tag{4}$$

where

$$\delta_{k,t}=\begin{cases}0 & \text{if} \quad k\neq t\\ 1 & \text{if} \quad k=t,\end{cases}$$

and

$$a\delta_{r,n}+\sum_{k=0}^{n} c_k(N)\binom{n}{k} \geqq 0. \tag{5}$$

It remains now to notice that (2) in fact is of the form of (4) or (5). Indeed, let $m(n)$ denote the number of the A_j, $1 \leqq j \leqq n$, which occur. Then $m(n) \leqq n$, and

$$I(B_{r,n}) = \delta_{r,m(n)}, \quad J_{k,n} = \binom{m(n)}{k}.$$

Thus (4) implies (2) if $m(n)=0$ and, for $m(n)>0$, (2) follows from (5). This completes the proof.

Some specific inequalities

All inequalities which follow will be proved by the method of polynomials. Some inequalities are known, but, in addition to their new proofs, they will be placed into a new context. A major result is the reduction of Bonferroni inequalities with $P(B_{r,n})$ to those with $P(B_{0,n})$. This idea will be presented through a concrete inequality.

Before we start with the list of inequalities, we wish to add that the method of polynomials can of course be used for proving identities which are linear in $P(B_{r,n})$ and in $S_{k,n}$. Namely, an identity can always be considered as a set of two inequalities ($f=g$ if, and only if, both $f \leqq g$ and $f \geqq g$).

We now give the inequalities in the form of theorems.

Theorem 2. *The sharpest inequality of the form*

$$c_k S_{k,n} \geqq S_{k+1,n}, \quad k \geqq 0, \tag{6}$$

is obtained by the choice $c_k=(n-k)/(k+1)$.

PROOF. By the method of polynomials, the sharpest form of (6) is obtained by finding the sharpest form of the inequalities

$$c_k(N)\binom{n}{k}p^k \geqq \binom{n}{k+1}p^{k+1}, \quad 0 \leqq p \leqq 1, \quad N \geqq n,$$

which can be simplified to

$$c_k(N)(k+1)/(n-k) \geqq p, \quad 0 \leqq p \leqq 1, \quad N \geqq n.$$

Evidently, the smallest value of $c_k(n)$, for which these inequalities hold, is equal to $(n-k)/(k+1)$. This completes the proof.

The inequality of Theorem 2 can also be written as

$$\binom{n}{k}^{-1} S_{k,n} \geqq \binom{n}{k+1}^{-1} S_{k+1,n},$$

which was established by FRECHET [3]. What is new here is the simple proof as well as the optimal property.

Theorem 3. *For each positive integer j,*

$$P(B_{0,n}) \leqq 1-\frac{2}{j+1}S_{1,n}+\frac{2}{j(j+1)}S_{2,n}. \tag{7}$$

Hence, the minimum in j leads to the estimate

$$P(B_{0,n}) \leqq 1-\frac{2}{j_0+1}S_{1,n}+\frac{2}{j_0(j_0+1)}S_{2,n}, \tag{8}$$

where j_0-1 is the integer part of $2S_{2,n}/S_{1,n}$.

There is an extensive literature on the inequality (8). It was first proved by Dawson and Sankoff [2], who have shown that (8) implies an earlier result of Chung and Erdős [1]. Recently, two optimal properties of (8) were obtained. Kwerel [7] has shown that the best upper bound on $P(B_{0,n})$ in terms of $S_{1,n}$ and $S_{2,n}$ is the inequality (8). A considerably simpler and shorter proof was given by one of us in [5] for the following result. *The best upper bound on $P(B_{0,n})$ in the form of*

$$P(B_{0,n}) \leqq c_0(n)+c_1(n)S_{1,n}+c_2(n)S_{2,n}$$

is the inequality (8). We give here *a new proof* for Theorem 3 by the method of polynomials. The essential part of the proof is Lemma 2 below, which will also be the essential part of Theorem 4.

Lemma 2. *For each pair (n, j) of positive integers, and for each real number p in the closed unit interval,*

$$1-2np/(j+1)+n(n-1)p^2/j(j+1) \geqq (1-p)^n. \tag{9}$$

Proof. For $j=1$, (9) is a special case of a polynomial inequality proved in [4]. Henceforth, we assume that $j\geqq 2$.

For $n=1$, (9) is obvious, while $n=2$ leads to a simple linear inequality. Let therefore $n\geqq 3$. Let us define

$$f_n(p) = 1-(1-p)^n-2np/(j+1)+n(n-1)p^2/j(j+1),$$

where $n\geqq 3$ and $j\geqq 2$. We have to show that $f_n(p)\geqq 0$ on the closed unit interval. Clearly, $f_n(0)=0$. Furthermore, $f_n(1)\geqq 0$ can easily be established by considering it as a quadratic form in n. Next we observe that $f_n'(p)=0$ means that the function $(1-p)^{n-1}$ intersects a straight line. Hence, $f_n'(p)$ can vanish at most at two points. Since $f_n'(0)>0$ (recall that $j\geqq 2$), the inequality $f_n(p)\geqq 0$ is nontrivial only if there are exactly two points $0<p_1<p_2<1$ where $f_n'(p)=0$. In particular, this implies that $f_n'(1)>0$, or, equivalently, that $n>j+1$. Since $f_n(p)$ now has a local minimum at p_2, it evidently suffices to show that $f_n(p_2)\geqq 0$. However, substitution of $f_n'(p_2)=0$ into the definition of $f_n(p_2)$ yields

$$j(j+1)f_n(p_2) = (n-1)(n-2)p_2^2-2(n-1)(j-1)p_2+j(j-1).$$

The right hand side as a function of p_2 is positive at zero and it has negative discriminant. Consequently, it is positive for all possible values of p_2. The lemma is thus established.

Theorem 3, of course, now follows, since the method of polynomials and (9) imply (7). For proving (8), it remains only to find the minimum of the right hand side of (7) in j for given values of $S_{1,n}$ and $S_{2,n}$. Since it is very simple, we omit the details.

Theorem 4. *For integers* $n \geqq 1$, $r \geqq 0$ *and* $j \geqq 1$ *with* $r \leqq n$,

$$P(B_{r,n}) \leqq S_{r,n} - \frac{2(r+1)}{j+1} S_{r+1,n} + \frac{(r+1)(r+2)}{j(j+1)} S_{r+2,n}.$$

The minimum of the right hand side is achieved if we choose j as the integer part of $1+(r+2)\,S_{r+2,n}/S_{r+1,n}$.

Proof. By the method of polynomials, we have to prove that, for $0 \leqq p \leqq 1$,

$$\binom{n}{r} p^r (1-p)^{n-r} \leqq \binom{n}{r} p^r - 2(r+1)\binom{n}{r+1}\frac{p^{r+1}}{(j+1)} + (r+1)(r+2)\binom{n}{r+2}\frac{p^{r+2}}{j(j+1)}.$$

However, after simplifying by $\binom{n}{r} p^r$, the above inequality reduces to (9) for the pair $(n-r, j)$. Thus Lemma 2, together with the method of polynomials, implies the actual inequality of the present theorem. Since the form of the right hand side of this inequality is the same as the one in (7), the value of j where the minimum is achieved can be obtained from Theorem 3. The proof is completed.

The preceding proof represents a general method of reducing a linear bound on $P(B_{r,n})$ to one on $P(B_{0,n-r})$. This possibility is due to the method of polynomials, by which a linear inequality of the form of (1) becomes a polynomial inequality, in which a simplification by a power of p means a reduction of r. This reduction method is one of the most significant consequence of the method of polynomials.

Let us conclude the paper with a remark. There are only a few inequalities which can be applied to $P(B_{r,n})$ with $r>0$. Most of them are concentrated around the method of inclusion and exclusion which have the following disadvantage: using a few binomial moments only, they are usually larger than one; hence these bounds become trivial. In our inequality, this is avoided by the arbitrary parameter j, which makes it possible to get nontrivial estimates with three binomial moments. For applications of inequalities of the form (1) to distribution problems of order statistics from dependent samples, see Chapter I in the book [6] by one of us.

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