An extension of a theorem of Baker

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1. Introduction

We say of the function $f(z)$ of the complex variable z that

(i) $f(z)$ belongs to class I, if $f(z)$ is rational,

(ii) $f(z)$ belongs to class II, if $f(z)$ is entire transcendental, and

(iii) $f(z)$ belongs to class III, if it is regular in the complex sphere punctured at $a, b (a \neq b)$ and has an essential singularity at b and a singularity at a (which may be a pole or an essential singularity) and if $f(z)$ omits the values a and b except possibly at a.

We can normalise the functions in class III, to make $a=0$ and $b=\infty$. In the following we shall consider only such normalised functions, whenever we deal with functions in class III.

The iterates $f_n(z)$ of $f(z)$ are defined inductively by

$$f_0(z) = z, \quad f_1(z) = f(z), \dots, f_{n+1}(z) = f(f_n(z)), \quad n = 0, 1, 2, \dots.$$

A *fixpoint* α of order n of $f(z)$ is a solution of $f_n(z)-z=0$, it has exact order n if

$$f_j(z) - z = 0$$

for $j=n$, but not for $j<n$.

In [1] BAKER proves the following theorem.

Theorem A. (BAKER) *If $f(z)$ belongs to class* II, *then $f(z)$ has fixpoints of exact order n, except for at most one value of n.*

Baker's proof of theorem A depends on a lemma of Pólya [see BAKER 1] and on Nevanlinna's second fundemental theorem.

Pólya's lemma, which is true for function in class II, does not apply to functions in class III and the second fundemental theorem needs to be restated in this case.

The object of this note is to extend theorem A to functions in class III. We prove

Theorem B. *If $f(z)$ belongs to class* III, *then $f(z)$ has an infinity of fixpoints of exact order n, for every positive integer n.*

The theory of iteration of functions belonging to classes I and II was developed by FATOU [3] and by JULIA [5]. RADSTRÖM [6] showed that the Fatou—Julia theory exists precisely for functions in classes I and II considered by them and for class III. He showed that the theory does not apply to any other class of functions.

2. Preliminaries

We use the following notations (c.f. 2., p. 88].
Let $f(z)$ be meromorphic in $r_0 \leqq |z| < \infty$, $r_0 > 0$ $n(t, a)$=number of roots of $f(z)=a$ in $r_0 < |t| \leqq r$

$$N(r, r_0; f, a) = N(r, a) = \int_{r_0}^{r} \frac{n(t, a)}{t} dt,$$

When $a=\infty$, we write $n(t, \infty)=n(t)$ to denote the number of poles in $r_0 < |t| \leqq r$ counted with due regard to multiplicity and write $N(r, \infty)=N(r, f)$.

We set

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\Theta})| \, d\Theta$$

$$m(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\Theta})-a} \right| d\Theta.$$

With these notations we can write Jensen's formula for a function meromorphic in $r_0 \leqq |z| < r$ as follows [c.f. 2., p. 88]

$$m(r, f)+N(r, f) = m\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f}\right)+O(\log r). \tag{1}$$

If we set

$$m(r, f)+N(r, f) = T(r, f), \tag{2}$$

then (1) becomes

$$T(r, f) = T\left(r, \frac{1}{f}\right)+O(\log r). \tag{3}$$

Then the first fundemental theorem of Nevanlinna takes the form

$$m(r, a)+N(r, a) = T(r)+O(\log r) \tag{4}$$

where $T(r)=T(r, f)$ and it is understood that we are referring to the region $r_0 \leqq |z| < \infty$.

In the usual development of the second fundemental theorem [see e.g. 4., p. 32] it is assumed that the function is meromorphic in $|z| < r$. If however, it is assumed that the function is meromorphic in $0 < r_0 \leqq |z| < \infty$ it can be seen that the proof carries through with only trivial adjustments. Thus we find that the theorem 2.1 of [4, p. 31] becomes the following.

Suppose that $f(z)$ is a nonconstant meromorphic function in $0 < r_0 < |z| < \infty$. Let $a_1, a_2, \ldots, a_q, q > 2$, be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\nu| \geqq \delta$ for $1 \leqq \mu \leqq \nu \leqq q$.

Then

$$m(r, f)+\sum_{\nu=1}^{q} m(r, a_\nu) \leqq 2T(r, f)-N_1(r)+S(r) \tag{5}$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f'), \tag{6}$$

and

$$S(r) = m\left(r, \frac{f'}{f}\right) + \sum_{\nu=1}^{q} m\left(r, \frac{f'}{f-a_\nu}\right) + O(\log r). \tag{7}$$

The proof is identical with that given in [4, p. 32] except that use of (4) instead of the usual form of the first fundemental theorem introduces the $O(\log r)$ term.

In [2, p. 94—98] it is shown that $m\left(r, \frac{f'}{f}\right)$ and hence $m\left(r, \frac{f'}{f-a}\right)$ is

$$O\{\max(\log^+ T(r, f), \log r)\} \quad as \quad r \to \infty \tag{8}$$

outside a set of r-intervals of finite measure.

Adding $N(r, f) + \sum_1^q N(r, a_\nu)$ to both sides of (5) and using (4) we obtain

$$(q-1)T(r, f) \leqq N(r, f) + \sum_1^q N(r, a_\nu) - N_1(r) + S(r) + O(\log r)$$

and hence

$$(q-1)T(r, f) \leqq \overline{N}(r, f) + \sum_1^q \overline{N}(r, a_\nu) + S_1(r) \tag{9}$$

where

$$\overline{N}(r, a_\nu) = \int_{r_0}^{r} \frac{\bar{n}(t, a_\nu)}{t}\, dt$$

and $\bar{n}(t, a_\nu)$ is the number of distinct roots of $f(z) = a_\nu$ in $r_0 < |t| \leqq r$ counted singly.

Moreover

$$S_1(r) = O\{\max(\log^+ T(r, f), \log r)\} \tag{10}$$

as $r \to \infty$ outside a set of intervals of finite measure.

We note [c.f. 2., p. 90] that if f is a function in class III, which must necessarily have an essential sigularity at ∞, then

$$\frac{T(r, f)}{\log r} \to \infty \quad as \quad r \to \infty.$$

Thus for functions in class III we can replace (10) by

$$S_1(r) = O(\log T(r, f)). \tag{11}$$

3. Proof *of theorem* B.

We consider the function

$$g(z) = \frac{f_n(z)}{z}.$$

Clearly

$$(12) \qquad T(r, g) = T(r, f_n) + O(\log r).$$

(13) $\begin{cases} \text{Now assume that } f_n(z)=z \text{ has only a finite number of solutions, which} \\ \text{are not also solutions of } f_j(z)=z \text{ for some } j<n, \text{ i.e. assume that } f(z) \\ \text{has finitely many fixpoints of exact order } n. \end{cases}$

Now using (9) for g in the region $r_0<|z|<\infty$ where $r_0>0$ is any fixed number, we have for our modified Nevanlinna theory

$$T(r, g) \leqq \overline{N}(r, g, 0) + \overline{N}(r, g, \infty) + \overline{N}(r, g, 1) + S(r, g)$$

where $S(r, g)=O(\log T(r, g))$ out side a set of r intervals of finite total length. Since

$$\overline{N}(r, g, 0) = 0, \quad \overline{N}(r, g, \infty) = 0$$

we have

$$T(r, g) \leqq \overline{N}(r, g, 1) + S(r, g) \leqq \sum_{1}^{n-1} N(r, f_j(z)-z, O) + S(r, g) + O(\log r) \leqq$$

(the $O(\log r)$ arises from the possibility that a finite number of solutions of $f_n(z)-z=0$ are of exact order n)

$$(14) \qquad \leqq \sum_{1}^{n-1} T(r, f_j, O) + S(r, g) + O(\log r) \leqq (1/2)T(r, f_n)$$

by (11) outside a set of r intervals of total finite length, provided we can show that

$$(15) \qquad \frac{T(r, f_j)}{T(r, f_n)} < \frac{1}{4n}, \quad \text{for} \quad j<n$$

out side a set of total finite length.

Assume for the moment that (15) is true. Then (14) contradicts (12). This contradiction arises only from our assumption (13). Hence (13) must be false, i.e. $f(z)$ has an infinity of fixpoints of exact order n. In fact the counting function $N(r)$ of such fixpoints must clearly satisfy $\overline{\lim}_{r\to\infty} \dfrac{N(r)}{T(r, f_n)} > 0$. This completes the proof of our theorem.

It remains now to prove (14). This we prove in the following lemma.

Lemma. *If n, p are positive integers and f is a function in class* III, *then for any $r_0>0$, we have*

$$\frac{T(r, f_{n+p})}{T(r, f_n)} > M_1,$$

where M_1 is an arbitrarily large constant, outside a set of r intervals of total finite length, and where $T(r, f)$ is as defined in (2).

Proof. Consider the equation

$$f_{n+p}(z) = a, \quad \text{where} \quad a \neq 0, \infty.$$

This is equivalent to

$$f_p(w) = a, \quad \text{at} \quad w_1, w_2, \ldots,$$

and

$$f_n(z) = w_i \quad (i = 1, 2, \ldots).$$

(Since $0, \infty$ are the only Picard values of f_p there is an infinity of w_i.)
Now by (4)

$$T(r, f_{n+p}) > \overline{N}(r, f_{n+p}, a) + O(\log r) \geqq \sum_{i=1}^{M} \overline{N}(r, f_n, w_i)$$

for any fixed M, say $> M_1 + 4$.
By (9) we have

$$T(r, f_{n+p}) > (M-3)T(r, f_4) - S(r) > (M-4)T(r, f_n),$$

outside a set of r intervals of total finite length,

i.e. $$\frac{T(r, f_{n+p})}{T(r, f_n)} > M - 4 > M_1$$

outside a set of r intervals of finite total length.

This proves the lemma, and hence the statements (14), (15) and theorem B.

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References

[1] I. N. Baker, The Existence of fixpoints of entire functions. *Math. Z.* **73** (1960), 280—284.
[2] L. Bieberbach, Theorie der Gewöhnlichen Differentialgleichungen, *Berlin* 1953.
[3] P. Fatou, Sur les équations fonctionelles, *Bull. Soc. Math. France* **47** (1919), 161—271 and **48** (1920), 33—94, 208—314.
[4] W. K. Hayman, Meromorphic functions, *Oxford*, 1964.
[5] G. Julia, Mémoire sur l'itération des fonctions rationelles. *J. Math. Pures Appl.*, Ser. 8. **1** (1918), 47—245.
[6] H. Rádström, On the iteration of analytic functions, *Math. Scand.* **1** (1953), 85—92.

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