Publ. Math. Debrecen
50 / 1-2 (1997), 135–143

The converse of a generalized Hölder inequality

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Abstract. Let (Ω, Σ, μ) be a measure space with two sets A, $B \in \Sigma$ such that $0 < \mu(A) < 1 < \mu(B) < \infty$, and k a fixed positive integer. Suppose that $\phi_1, \dots, \phi_k$, are arbitrary bijections of $(0, \infty)$. The main result says that if

$$\int_\Omega x_1 \cdot \ldots \cdot x_k d\mu \le \phi_1^{-1} \left(\int_{\Omega(x_1)} \phi_1 \circ x_1 \, d\mu \right) \cdot \ldots \cdot \phi_k^{-1} \left(\int_{\Omega(x_k)} \phi_k \circ x_k \, d\mu \right)$$

for all μ–integrable nonnegative step functions $x_1, \dots, x_k$, then $\phi_1, \dots, \phi_k$ must be conjugate power functions (here $\Omega(x) = \{\omega \in \Omega : x(\omega) \neq 0\}$.

Introduction

For a measure space (Ω, Σ, μ) denote by $\boldsymbol{S} = \boldsymbol{S}(\Omega, \Sigma, \mu)$ the linear space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$, and by $\boldsymbol{S}_+ = \boldsymbol{S}_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $x \in \boldsymbol{S}(\Omega, \Sigma, \mu)$. For an arbitrary bijection $\phi : (0, \infty) \to (0, \infty)$ the functional $\boldsymbol{p}_{\boldsymbol{\phi}} : \boldsymbol{S} \to \mathbb{R}_+$ ($\mathbb{R}_+ := [0, \infty)$) given by

$$\boldsymbol{p}_{\boldsymbol{\phi}}(x) := \begin{cases} \phi^{-}1 \left(\int_{\Omega(x)} \phi \circ |x| d\mu \right) & \text{if} \quad \mu(\Omega(x)) > 0 \\ 0 & \text{if} \quad \mu(\Omega(x)) = 0 \end{cases}, \quad x \in \boldsymbol{S}(\Omega, \Sigma, \mu),$$

where $\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}$, is well defined (cf. [3]).

Mathematics Subject Classification: 1991, *Primary*: 26A51, 26D15, 39B72.
Key words and phrases: measure space, integrable step functions, the converse of Hölder inequality, conjugate functions.

Note that for $\phi(t) := \phi(1)t^p$, $t > 0$, where $p \in \mathbb{R} \setminus \{0\}$ is arbitrary fixed, we have

$$\boldsymbol{p_\phi}(x) = \left(\int_{\Omega(x)} |x|^p d\mu\right)^{\frac{1}{p}}, \qquad x \in \boldsymbol{S}(\Omega, \Sigma, \mu), \qquad \mu(\Omega(x)) > 0.$$

and for $p \geq 1$ the functional $\boldsymbol{p_\phi}$ becomes the $\boldsymbol{L}^p$-norm. Let k be a fixed positive integer, and $\phi_1, \dots, \phi_k$ bijections of $(0, \infty)$. Suppose that the inequality

$$\int_\Omega x_1 \cdot \ldots \cdot x_k \, d\mu \leq \boldsymbol{p}_{\boldsymbol{\phi}_1}(x_1) \cdot \ldots \cdot \boldsymbol{p}_{\boldsymbol{\phi}_k}(x_k), \qquad x_1, \dots, x_k \in \boldsymbol{S}_+,$$

holds true. We prove that if $\phi_1, \dots, \phi_k$ are *multiplicatively conjugate*, i.e. there is a constant $c > 0$ such that

$$\phi_1^{-1}(t)\phi_2^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t) = ct, \qquad t > 0,$$

and the measure space (Ω, Σ, μ) is not trivial, then $\phi_1, \dots, \phi_k$ must be power functions. The main purpose of this paper is to prove that if there are two sets A, $B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty,$$

then $\phi_1, \dots, \phi_k$ are multiplicatively conjugate power functions.

These results are the converses of a known generalized Hölder inequality (cf. Hardy–Littlwood–Pólya [1], p. 140, Theorem 188, also p. 21, Theorem 10). An analogous result for $k = 2$, under a little stronger assumptions, has been proved in [6].

The relevant results for the reversed Hölder inequality are also given.

1. Auxiliary results

A crucial role plays the following

Lemma 1 ([5]). *Let a and b be real numbers such that*

$$0 < \min\{a, b\} < 1 < a + b.$$

If a function $f : (0, \infty) \to \mathbb{R}_+$ satisfies the inequality

$$af(s) + bf(t) \leq f(as + bt), \qquad s, t > 0,$$

then $f(t) = f(1)t$, $(t > 0)$.

For the reversed inequality we have the following

Lemma 2 ([4]). *Let a and b be real numbers such that*

$$0 < \min\{a,b\} < 1 < a+b.$$

If a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded in a neighbourhood of 0, $f(0) = 0$ and

$$f(as+bt) \le af(s) + bf(t), \qquad s,\ t \ge 0,$$

then $f(t) = f(1)t$, $(t \ge 0)$.

We need also the following result on a simultaneous system of two functional equations.

Lemma 3 ([2]). *Let a, b, α, β be positive real numbers and suppose that $h : (0,\infty) \to (0,\infty)$ is continuous at least at one point and satisfies the system of functional equations*

$$h(at) = \alpha h(t), \qquad h(bt) = \beta h(t), \qquad t > 0.$$

If $a \ne 1$ and $\frac{\log b}{\log a}$ is irrational then there exists a $q \in \mathbb{R}$ such that $h(t) = h(1)t^q$, for all $t > 0$.

2. The converse of generalized Hölder's inequality for multiplicatively conjugate functions

We start this section with the following

Theorem 1. *Let (Ω, Σ, μ) be a measure space with two disjoint sets A, $B \in \Sigma$ of finite and positive measure, and k a fixed positive integer. If $\phi_1, \dots, \phi_k : (0,\infty) \to 0,\infty)$ are bijections such that for a positive c,*

$$\phi_1^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t) = ct, \qquad t > 0, \tag{1}$$

and

$$\int_\Omega x_1 \cdot \ldots \cdot x_k \, d\mu \le \boldsymbol{p}_{\boldsymbol{\phi}_1}(x_1) \cdot \ldots \cdot \boldsymbol{p}_{\boldsymbol{\phi}_k}(x_k), \qquad x_1, \dots, x_k\ in \boldsymbol{S}_+, \tag{2}$$

then $\phi_1, \dots, \phi_k$ are conjugate power functions, i.e. there are $q_1, \dots, q_k \in \mathbb{R}$, $q_1, \dots, q_k \ge 1$, such that

$$\phi_i(t) = \phi_i(1)t^{q_i}, \qquad t > 0;\ \ i = 1, \dots, k,$$

and

$$q_1^{-1} + \ldots + q_k^{-1} = 1.$$

Proof. For $k = 1$ we have $\phi_1^{-1}(t) = ct$, $t > 0$, and the theorem is obvious. A formal proof in the general case $k \geq 2$ requires quite complicated notation. Since the idea of the proof in the general case is exactly the same as in the case $k = 3$, we give the detailed argument for $k = 3$. For the simplicity of notations we put $\phi := \phi_1$, $\psi := \phi_2$, $\gamma := \phi_3$. By χ_A we denote the characteristic function of a set A. Put $a := \mu(A)$, $b := \mu(B)$. Setting in inequality (2) arbitrary x, y, $z \in \boldsymbol{S}_+$ of the form:

$$x := x_1\chi_A + x_2\chi_B, \quad y := y_1\chi_A + y_2\chi_B, \quad z := z_1\chi_A + z_2\chi_B, \\ x_i,\ y_i,\ z_i > 0,$$

and making use of the definition of $\boldsymbol{p_\phi}$, we get the inequality

$$ax_1y_1z_1 + bx_2y_2z_2 \\ \leq \phi^{-1}\left(a\phi(x_1) + b\phi(x_2)\right)\psi^{-1}\left(a\psi(y_1)\ +\ b\psi(y_2)\right)\gamma^{-1}\left(a\gamma(y_1) + b\gamma(y_2)\right)$$

for all x_i, y_i, $z_i > 0$. Replacing here x_i, y_i, and z_i, respectively by $\phi^{-1}(x_i)$, $\psi^{-1}(y_i)$, and $\phi^{-1}(z_i)$, $i = 1$, 2, we obtain

$$\begin{aligned} &a\phi^{-1}(x_1)\psi^{-1}(y_1)\gamma^{-1}(z_1) + b\phi^{-1}(x_2)\psi^{-1}(y_2)\gamma^{-1}(z_2) \\ &\quad\leq \phi^{-1}(ax_1 + bx_2)\psi^{-1}(ay_1 + by_2)\gamma^{-1}(az_1 + bz_2) \end{aligned} \tag{3}$$

for all x_1, x_2, z_1, y_1, y_2, $z_2 > 0$. Similarly, setting in (2)

$$x := x_1\chi_A, \qquad y := y_1\chi_A, \qquad z := z_1\chi_A, \qquad x_1,\ y_1,\ z_1 > 0,$$

we obtain

$$a\phi^{-1}(x_1)\psi^{-1}(y_1)\gamma^{-1}(z_1) \leq \phi^{-1}(ax_1)\psi^{-1}(ay_1)\gamma^{-1}(az_1), \\ x_1,\ y_1,\ z_1 > 0.$$

From (1) we have

$$\psi^{-1}(t)\gamma^{-1}(t) = \frac{ct}{\phi^{-1}(t)}, \qquad t > 0. \tag{4}$$

Hence, taking $z_1 := y_1$, we get

$$\frac{\phi^{-1}(ay_1)}{\phi^{-1}(y_1)} \leq \frac{\phi^{-1}(ax_1)}{\phi^{-1}(x_1)}, \qquad x_1,\ y_1 > 0.$$

This implies that the function $t \to \frac{\phi^{-1}(t)}{\phi^{-1}(a^{-1}t)}$ is constant in $(0, \infty)$ and, consequently, we have

$$\frac{\phi^{-1}(a^{-1}x_1)}{\phi^{-1}(a^{-1}y_1)} = \frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)}, \qquad x_1,\ y_1 > 0. \tag{5}$$

In the same way we show that

$$\frac{\phi^{-1}(b^{-1}x_2)}{\phi^{-1}(b^{-1}y_2)} = \frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)}, \qquad x_2,\ y_2 > 0. \tag{6}$$

From (3) and (4) we obtain

$$ay_1\frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)} + by_2\frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)} \le (ay_1 + by_2)\frac{\phi^{-1}(ax_1 + bx_2)}{\phi^{-1}(ay_1 + by_2)}.$$

Replacing here x_1, x_2, y_1, y_2 resp. by $a^{-1}x_1$, $b^{-1}x_2$, $a^{-1}y_1$, $b^{-1}y_2$ we get

$$y_1\frac{\phi^{-1}(a^{-1}x_1)}{\phi^{-1}(a^{-1}y_1)} + y_2\frac{\phi^{-1}(b^{-1}x_2)}{\phi^{-1}(b^{-1}y_2)} \le (y_1 + y_2)\frac{\phi^{-1}(x_1 + x_2)}{\phi^{-1}(y_1 + y_2)}.$$

Now from (5) and (6) we obtain the inequality

$$y_1\frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)} + y_2\frac{\phi^{-1}(x_2)}{\phi^{-1}(y_2)} \le (y_1 + y_2)\frac{\phi^{-1}(x_1 + x_2)}{\phi^{-1}(y_1 + y_2)}, \tag{7}$$

valid for all x_1, x_2, y_1, $y_2 > 0$. Setting

$$F(t) := \psi^{-1}(t)\gamma^{-1}(t), \qquad t > 0,$$

and making again use of (4) we can write this inequality in the form

$$\phi^{-1}(x_1)F(y_1) + \phi^{-1}(x_2)F(y_2) \le \phi^{-1}(x_1 + x_2)F(y_1 + y_2),$$
$$x_1,\ x_2,\ y_1,\ y_2 > 0.$$

Now we can prove that ϕ and F are homeomorphisms in $(0,\infty)$. In view of (1) it is sufficient to show that either ϕ^{-1} or F is increasing in $(0,\infty)$. Suppose for instance that F is not increasing in $(0,\infty)$. Thus $F(y_1) > F(y_1 + y_2)$ for some positive y_1, y_2 and the last inequality implies that $\phi^{-1}(x_1) < \phi^{-1}(x_1 + x_2)$ for all x_1, $x_2 > 0$, i.e. that ϕ^{-1} is increasing in $(0,\infty)$.

From (7), by induction, we obtain

$$y_1\frac{\phi^{-1}(x_1)}{\phi^{-1}(y_1)} + \ldots + y_n\frac{\phi^{-1}(x_n)}{\phi^{-1}(y_n)} \le (y_1 + \ldots + y_n)\frac{\phi^{-1}(x_1 + \ldots + x_n)}{\phi^{-1}(y_1 + \ldots + y_n)},$$

for all positive $x_1, \ldots, x_n$; $y_1, \ldots, y_n$ and $n \in \mathbb{N}$. Setting in this inequality $x_1 = \ldots = x_n := s$; $y_1 = \ldots = y_n := t$, we get

$$\frac{\phi^{-1}(nt)}{\phi^{-1}(t)} \le \frac{\phi^{-1}(ns)}{\phi^{-1}(s)}, \qquad s,\ t > 0;\ n \in \mathbb{N}.$$

It follows that for every $n \in \mathbb{N}$ the function $t \to \frac{\phi^{-1}(nt)}{\phi^{-1}(t)}$, $t > 0$, is constant. Hence for every $n \in \mathbb{N}$ there is $\alpha_n > 0$ such that

$$\phi^{-1}(nt) = \alpha_n \phi^{-1}(t), \qquad t > 0.$$

Taking $n = 2$ and $n = 3$ we see that $h := \phi^{-1}$ satisfies the system of functional equations

$$h(2t) = \alpha h(t), \qquad h(3t) = \beta h(t), \qquad t > 0,$$

where $\alpha := \alpha_2$, $\beta := \alpha_3$. Since h is continuous and $\log 3/\log 2$ is irrational, Lemma 3 implies that there is a $q_1 \in \mathbb{R}$ such that $h(t) = h(1)t^{1/q_1}$, $t > 0$, i.e.

$$\phi_1^{-1}(t) = \phi^{-1}(t) = \phi^{-1}(1)t^{1/q_1}, \qquad t > 0.$$

By the monotonicity of ϕ^{-1} we have $q_1 > 0$.

In the same way one can show that $\phi_i^{-1}(t) = \phi_i^{-1}(1)t^{1/q_i}$ $(t > 0)$ for some $q_i > 0$, and $i = 2, \dots, k$. By (1) we have

$$q_1^{-1} + \ldots + q_k^{-1} = 1,$$

and consequently, $q_i > 1,\ i = 1, \dots, k$. This completes the proof. □

Remark 1. Note that carrying out the argument for arbitrary $k \in \mathbb{N}$, $k \geq 3$, we can define the function F as follows

$$F(t) := \phi_2^{-1}(t) \cdot ... \cdot \phi_k^{-1}(t), \qquad t > 0.$$

Similarly, applying Lemma 2, we can prove

Theorem 2. *Let (Ω, Σ, μ) be a measure space with two disjoint sets of finite and positive measure. If $\phi_1, \dots, \phi_k : (0, \infty) \to (0, \infty)$ are bijections such that for some positive c:*

$$\phi_1^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t) = ct, \qquad t > 0,$$

and

$$\boldsymbol{p}_{\boldsymbol{\phi}_1}(x_1) \cdot \ldots \cdot \boldsymbol{p}_{\boldsymbol{\phi}_k}(x_k) \leq \int_\Omega x_1 \cdot \ldots \cdot x_k \, d\mu, \qquad x_1, \dots, x_k \in \boldsymbol{S}_+,$$

then $\phi_1, \dots, \phi_k$ are conjugate power functions i.e. there are $q_1, \dots, q_k \in \mathbb{R} \setminus \{0\}$ such that

$$\phi_i(t) = \phi_i(1)t^{q_i}, \qquad t > 0;\ \ i = 1, \dots, k,$$

and

$$q_1^{-1} + \ldots + q_k^{-1} = 1.$$

3. The main theorem

The main goal of this paper is to prove the following

Theorem 3. *Suppose that* (Ω, Σ, μ) *is a measure space with two sets* A, $B \in \Sigma$ *such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

If $\phi_1, \dots, \phi_k : (0,\infty) \to (0,\infty)$ *are arbitrary bijections such that*

$$\int_\Omega x_1 \cdot \ldots \cdot x_k \, d\mu \le \boldsymbol{p}_{\boldsymbol{\phi}_1}(x_1) \cdot \ldots \cdot \boldsymbol{p}_{\boldsymbol{\phi}_k}(x_k), \qquad x_1, \dots, x_k \in \boldsymbol{S}_+,$$

then $\phi_1, \dots, \phi_k$ *are conjugate power functions i.e. there are* $q_1, \dots, q_k \in \mathbb{R}$, $q_1, \dots, q_k \ge 1$, *such that*

$$\phi_i(t) = \phi_i(1) t^{q_i}, \qquad t > 0;\ i = 1, \dots, k,$$

and

$$q_1^{-1} + \ldots + q_k^{-1} = 1.$$

Proof. Define $f : (0,\infty) \to (0,\infty)$ by

$$f(t) := \phi_1^{-1}(t)\phi_2^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t), \qquad t > 0,$$

and put $a := \mu(A)$ and $b := \mu(B \setminus A)$. Setting in the assumed inequality $x_i := s_i \chi_A + t_i \chi_{B\setminus A} \in \boldsymbol{S}_+$ $(i = 1, \dots, k)$, we obtain,

$$a\phi_1^{-1}(s_1)\phi_2^{-1}(s_2) \cdot \ldots \cdot \phi_k^{-1}(s_k) + b\phi_1^{-1}(t_1)\phi_2^{-1}(t_2) \cdot \ldots \cdot \phi_k^{-1}(t_k)$$
$$\le \phi_1^{-1}(as_1 + bt_1)\phi_2^{-1}(as_2 + bt_2) \cdot \ldots \cdot \phi_k^{-1}(as_k + bt_k)$$

for all positive $s_1, \dots, s_k$; $t_1, \dots, t_k$. Taking here $s_1 = s_2 = \ldots = s_k := s$; $t_1 = t_2 = \ldots = t_k := t$ gives

$$af(s) + bf(t) \le f(as + bt), \qquad s,\ t > 0.$$

Since $0 < a < 1 < a + b$ it follows by Lemma 1 that $f(t) = f(1)t$, $(t > 0)$. Thus, by the definition of f, the functions ϕ_i, $i = 1, \dots, k$, are multiplicatively conjugate and our result is a consequence of Theorem 1. □

Remark 2. In Theorem 3 (as well as in Theorem 1), if $k \ge 2$ then $q_i > 1$ for all $i = 1, \dots, k$. If $k = 1$ then $q_1 = 1$, and the basic Hölder inequality (2) becomes an equality.

Remark 3. Theorem 3 generalizes the main result of a paper [6] where the case $k = 2$ is considered, and the functions ϕ_1, ϕ_2 are assumed to be bijections of $\mathbb{R}_+ = [0, \infty)$.

Remark 4. If we assume that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, and

$$\phi(0) = 0,$$

then the definition of the functional $\boldsymbol{p_\phi} : \boldsymbol{S}_+ \to \mathbb{R}_+$ simplifies to the following formula

$$\boldsymbol{p_\phi}(x) := \phi^{-1}\left(\int_\Omega \phi \circ x \, d\mu\right), \qquad x \in \boldsymbol{S}_+ \ .$$

Using this remark, and applying Lemma 2 and Theorem 2, we can prove

Theorem 4. *Suppose that (Ω, Σ, μ) is a measure space with two sets A, $B \in \Sigma$ such that $0 < \mu(A) < 1 < \mu(B) < \infty$. If $\phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ are bijections such that $\phi_i(0) = 0$, $i = 1, \dots, k$, the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula*

$$f(t) := \phi_1^{-1}(t) \cdot \ldots \cdot \phi_k^{-1}(t), \qquad t \geq 0,$$

is bounded in a neighbourhood of 0 *and*

$$\boldsymbol{p}_{\boldsymbol{\phi}_1}(x_1) \cdot \ldots \cdot \boldsymbol{p}_{\boldsymbol{\phi}_k}(x_k) \leq \int_\Omega x_1 \cdot \ldots \cdot x_k \, d\mu, \qquad x_1, \dots, x_k \in \boldsymbol{S}_+,$$

then $\phi_1, \dots, \phi_k$ are conjugate power functions, i.e. there are $q_1, \dots, q_k \in \mathbb{R} \setminus \{0\}$, such that

$$\phi_i(t) = \phi_i(1) t^{q_i}, \qquad t \geq 0; \quad i = 1, \dots, k,$$

and

$$q_1^{-1} + \ldots + q_k^{-1} = 1.$$

Remark 5. In Theorem 4 (and Theorem 2), if $k \geq 2$ then at least one of the numbers q_i, $i = 1, \dots, k$, is negative, and the relevant power function ϕ_i is decreasing in $(0, \infty)$. If $k = 1$ then $q_1 = 1$, and the assumed reversed Hölder inequality becomes an equality.

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(Received April 2, 1996)