On ordered Jordan algebras

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Abstract: In [2] FUCHS has shown that the concept of lattice ordered rings introduced by BIRKHOFF and PIERCE can be extended to some extent to nonassociative rings, and proposed a study of ordered Jordan algebras. In this paper we have characterized finite dimensional semisimple fully ordered Jordan algebras and proved that any finite dimensional semisimple fully ordered Jordan algebra is either an exceptional Jordan division algebra or a Jordan division algebra of symmetric elements of an associative division algebra.

Let A be an associative algebra over a field F of char. $\neq 2$. Define a binary operation by $x \cdot y = \frac{1}{2}(xy+yx)$. Then $A^+ = (A, +, \cdot, F)$ is a commutative algebra satisfying the identities

(1) $$(x, y, x^2) = 0$$

(2) $$(x, y, x) = 0.$$

An algebra satisfying (1) and (2) is called a *Jordan algebra*. In this paper we shall consider only finite dimensional Jordan algebras which have been studied by N. JACOBSON and others [3].

The following theorem characterises special Jordan division algebras.

Theorem 1 [3, p. 210] *A special finite dimensional commutative Jordan division algebra over F is isomorphic to one of the following types*

(a) *The Jordan algebra of a nondegenerate symmetric bilinear form f on a finite dimensional vector space B/K, $\dim B/K > 1$ where K is a finite dimensional field extension of F and $f(x, x) = 1$ has no solution.*

(b) *A Jordan algebra A^+ where A is a finite dimensional associative division algebra.*

(c) *A Jordan algebra $S(A, \theta)$ where A is a finite dimensional associative division algebra with an involution θ.*

A partially ordered Jordan algebra is said to be fully ordered if $(A, +)$ is a fully ordered group. For results on ordered rings one can see [2].

If J is a fully ordered *finite dimensional* semisimple Jordan algebra over a field F, then J is a direct sum of simple ideals $\{J_i\}_{i=1}^n$, and each J_i has an identity [3]. Since J is *fully ordered,* J is a simple algebra with identity 1. Moreover it is easy to see that 1 is the only nonzero idempotent in J.

Now let $x \in J$. If x is not nilpotent then $B = \{a_1 x + \dots + a_n x^n | a_i \in F\}$ is an associative subalgebra which is not nil. Therefore $1 \in B$ and x is invertible. Let N be the set of all nilpotent elements of A. For any $a\in N$, $b\in J$, $\{aba\}=2(ba)a-ba^2$ is nilpotent and hence $2((ba)a)a-(ba^2)a=2((ba)a)a-(ba)a^2=\{a(ba)a\}\in N$. Thus $ba\in N$ and N is a nil ideal of J [2]. Since J is semisimple, $N=(0)$. Hence we have proved the following lemma.

Lemma 2. *Any commutative semisimple fully ordered Jordan algebra is a fully ordered division algebra without zero divisors.*

Let J be a fully ordered commutative semisimple Jordan algebra over a field F, where F is a finite extension of rationals. By Lemma 2, $1\in J$ and hence F has an order induced by the order on J, namely $a\in F$, $a\geqq 0$ if and only if $a.1\geqq 0$ in J.

Since $Q\subseteqq F$, induced order on Q coincides with the natural order and so F is an Archimedean field. Moreover J is Archimedean. For, if there exist $a, b\in J$, $a\neq 0$ such that $na<b$ for all n. Then $n(ab^{-1})<1$ for all n. Therefore without loss of generality we can assume that there is a $y>0$ such that $ny<1$ for all n. Let $f(x)=a_0+a_1x+\ldots+a_mx^m\in F[x]$ be the minimal polynomial of y. Then $a_0\neq 0$ and for all integer n,

$$n|a_0|\cdot 1 \leqq |n|\,|a_1y_1+\ldots+a_my^m| \leqq (|a_1|+\ldots+|a_m|)\cdot 1.$$

This implies that F is not Archimedean which is a contradiction. Hence J is an ordered field [2]. This establishes following:

Theorem 3. *Let J be a finite dimensional semisimple Jordan algebra over a field F, where F is a finite extension of rationals. Then J is fully ordered if and only if J is an ordered field.*

In the proof of above theorem we only made use of the fact that F is Archimedean. So the same proof will hold if F is replaced by the field R of real numbers.

Theorem 4. *Any finite dimensional semisimple fully ordered Jordan algebra is either an exceptional division algebra or a Jordan algebra of symmetric elements of an associative division algebra.*

PROOF. Let J be a finite dimensional semisimple fully ordered Jordan algebra. Then by Lemma 2 J is a fully ordered division algebra. If J is special, by Theorem 1 it is isomorphic to one of the types (a), *(b)* or *(c)*.

Suppose J is of type (a), that is J is a Jordan algebra of a nondegenerate symmetric bilinear form f on a finite dimensional vector space V, dim $V>1$. Without loss of generality we can assume that $V=V_n(F)$ and for any $x=(x_1, \ldots, x_n)$, $y=(y_1, y_2, \ldots, y_n)$ in $V_n(F)$ $f(x, y)=a_1x_1y_1+a_2x_2y_2+\ldots+a_nx_ny_n$. For $v_i=(0, \ldots, \ldots, 1, 0, \ldots, 0)\in V$, $i=1, 2, \ldots, n$, $v_i^2=a_i\cdot 1$. So $a_i\cdot 1\geqq 0$ for all $i=1, 2, \ldots, n$. Since f is non degenerate, $a_i\cdot 1>0$ for all $i=1, 2, \ldots, n$. But for $i\neq j$, $|v_i|\cdot|v_j|=$ $=|v_iv_j|=0$. Hence, either $a_i\cdot 1=0$ or $a_j\cdot 1=0$ and so dim $V=1$. But this is contrary to our hypothesis and so J cannot be type (a).

Now suppose $J=A^+$ for some finite dimensional associative division algebra A. Since J is fully ordered, A itself is fully ordered. For, if not then there exist $b>0$, $a>0$ such that $ba<0, a\cdot b\geqq 0$. So $ab>0$. Consider ab^{-1}. If $ab^{-1}\geqq 0$ then $b\cdot ab^{-1}\geqq 0$ and $ba\cdot b^{-1}\leqq 0$ and hence $ab=-ba$ and $a\cdot b=0$. Similarly if $ab^{-1}<0$ then $a\cdot b=0$. Hence in any case J has zero divisors, a contradiction. Therefore, A is a fully ordered associative division algebra. But any finite dimensional associative fully ordered division algebra is a field. So $a \cdot b = ab$ and J is associative. This completes the proof.

It is known that any semisimple commutative power associative algebra of characteristic zero is a Jordan algebra and hence Theorem 3 holds for such an algebra.

Now if A is a finite dimensional semisimple power associative, flexible algebra over a field F which is a finite extension of rationals. Then OEHMKE [5] has shown that A has an identity and A is a direct sum of simple algebras. Hence if A is fully ordered, then A must be simple with identity. Thus A^+ is a Jordan algebra [4]. The full order of A easily carries over to A^+ and hence A^+ is a semisimple commutative Jordan algebra. Thus as seen in the proof of Theorem 3, A^+ is Archimedean. Therefore, A is an Archimedean fully ordered ring and hence associative [2]. We have thus proved the following theorem.

Theorem 5. *Any finite dimensional semisimple power associative flexible algebra A over an algebraic number field F is fully ordered if and only if it is an ordered field.*

References

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