On a theorem of Daróczy, Lajkó and Székelyhidi

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In [2] DARÓCZY, LAJKÓ and SZÉKELYHIDI proved

Theorem 1. *Let P denote the set of positive elements of an ordered field F and let G be an additive abelian group. For $t\in P$ and $f\colon P\to G$ define $\delta_t f\colon P\to G$ by $\delta_t f(x)=f(xt)-f(x)$ for all $x\in P$. Then $f\colon P\to G$ satisfies*

$$2\delta_t f\left(\frac{x+y}{2}\right)=\delta_t f(x)+\delta_t f(y)\quad \textit{for all}\quad x,y,t\in P$$

if and only if there exist $\alpha, m\colon P\to G$ such that

$$f(x)=\alpha(x)+m(x)$$

$$2\alpha\left(\frac{x+y}{2}\right)=\alpha(x)+\alpha(y)$$

and

$$m(xy)=m(x)+m(y)\quad \textit{for all}\quad x,y\in P.$$

This theorem has found applications in solving functional equations in [1], [2] and [4]. The aim of this paper is to generalize theorem 1 (see theorem 3 below). The main tools will be some results of DJOKOVIĆ [3].

Thoughout this paper $(G,+)$ denotes an abelian group. We say that G admits division by the positive integer n provided that for every $x\in G$ there is a unique $y\in G$ such that $x=ny$ in which case we write $y=x/n$.

If S is a non empty set, G^S denotes the set of all mappings of S into G. If for $f,g\in G^S$ we define $f+g$ by $(f+g)(x)=f(x)+g(x)$ for all $x\in S$ then G^S becomes an abelian group.

Let

$$L(S,G)=\{E\colon G^S\to G^S \mid E(f+g)=E(f)+E(g) \text{ for all } f,g\in G^S\}.$$

If for $E,E'\in L(S,G)$ we define $E+E'$ and EE' by $(E+E')(f)=E(f)+E'(f)$ and $(EE')(f)=E(E'f)$ for all $f\in G^S$ then $L(S,G)$ becomes a ring with identity I-the identity map of G^S. For $E\in L(S,G)$ and $f\in G^S$ we usually write Ef instead of $E(f)$. For $E\in L(S,G)$ we let $E^2=EE$, $E^3=E^2E$, etc. Notice that if G admits division by n then so do G^S and $L(S,G)$.

Suppose $(S,+)$ is an abelian semigroup. For $y\in S$ define $\Delta_y\in L(S,G)$ by

$$\Delta_y f(x)=f(x+y)-f(x)$$

for all $x \in S$ and $f \in G^S$. Notice that

$$\Delta_y^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x+ky)$$

for all $x, y \in S$, all $f \in G^S$ and every positive integer n.

If k is a positive integer let $A^k(S, G)$ denote the set of all $A\colon S^k \to G$ such that

$$A(s_1+s_1', s_2, \dots, s_k) = A(s_1, s_2, \dots, s_k) + A(s_1', s_2, \dots, s_k)$$

and

$$A(s_{i_1}, s_{i_2}, \dots, s_{i_k}) = A(s_1, s_2, \dots, s_k)$$

for all $s_1, s_1', s_2, \dots, s_k \in S$ and for every permutation $(i_1, i_2, \dots, i_k)$ of $(1, 2, \dots, k)$. For $A \in A^k(S, G)$ we let $A^*(s) = A(s, s, \dots, s)$ for all $s \in S$. We need the following theorem of Djoković [3] (also see [5], [6] and [7]).

Theorem 2. *Suppose G admits division by $(n-1)!$ and $f\colon S \to G$. Then the following are equivalent:*

(i) $\Delta_y^n f(x) = 0$ *for all* $x, y \in S$,

(ii) $\Delta_{y_1} \Delta_{y_2} \dots \Delta_{y_n} f(x) = 0$ *for all* $x, y_1, \dots, y_n \in S$,

(iii) *there exist* $A_0 \in G$ *and* $A_k \in A^k(S, G)$, $1 \leqq k \leqq n-1$,

such that $f(x) = A_0 + \sum_{k=1}^{n-1} A_k^*(x)$ *for all* $x \in S$.

Corollary. *Suppose G admits division by $(n-1)!$ and $f\colon S \to G$ such that $\Delta_y^n f(x) = 0$ for all $x, y \in S$. Then there exists a constant $c \in G$ such that*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(ky) = c \quad \textit{for all} \quad y \in S.$$

Proof. By theorem 2 there exist $A_0 \in G$ and $A_j \in A^j(S, G)$, $1 \leqq j \leqq n-1$, such that

$$f(x) = A_0 + \sum_{j=1}^{n-1} A_j^*(x) \quad \text{for all} \quad x \in S.$$

Hence, for all $y \in S$,

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(ky) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \left(A_0 + \sum_{j=1}^{n-1} A_j^*(ky)\right) =$$

$$= \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^k\right) A_0 + \sum_{j=1}^{n-1} \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^k k^j\right) A_j^*(y).$$

Since $\sum_{k=1}^{n} \binom{n}{k} (-1)^k = -1$, to complete the proof it suffices to show that $\sum_{k=1}^{n} \binom{n}{k} (-1)^k k^j = 0$ whenever $1 \leqq j \leqq n-1$.

Now if Z denotes the integers, $1 \leqq j \leqq n-1$ and $g: Z \to Z$ is defined by $g(x) = x^j$ for $x \in Z$ then, by theorem 2, $\Delta_y^n g(x) = 0$ for all $x, y \in Z$. That is

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k g(x+ky) = 0 \quad \text{for all} \quad x, y \in Z.$$

Putting $x=0$ and $y=1$ we find $\sum_{k=1}^{n} \binom{n}{k} (-1)^k k^j = 0$.

Let P be an additive abelian semigroup in which an associative multiplication (xy) is defined which is distributive over addition. Also assume P has a multiplicative identity e. More explicitly we are assuming that for all $x, y, z \in P$,

$$(x+y)+z = x+(y+z), \quad x+y = y+x$$

$$(xy)z = x(yz), \quad xe = x = ex,$$

$$x(y+z) = xy+xz \quad \text{and} \quad (x+y)z = xz+yz.$$

For $t \in P$ define $\delta_t \in L(P, G)$ by

$$\delta_t f(x) = f(xt) - f(x)$$

for all $x \in P$ and $f \in G^P$.

The main result of this paper is

Theorem 3. *Suppose n is a positive integer, G admits division by $(n-1)!$ and $f: P \to G$. Then*

$$\Delta_y^n \delta_t f(x) = 0 \quad \textit{for all} \quad x, y, t \in P \tag{1}$$

if and only if there exist $g, h: P \to G$ such that

$$f = g + h, \tag{2}$$

$$\Delta_y^n g(x) = 0 \tag{3}$$

and

$$h(xy) = h(x) + h(y) \quad \textit{for all} \quad x, y \in P. \tag{4}$$

Proof. Suppose (1) holds. According to the corollary there exists $c(t) \in G$ such that

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k (\delta_t f)(kx) = c(t) \quad \text{for all} \quad x, t \in P.$$

On the other hand if we let

$$\varphi(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k f(kx) \quad \text{for} \quad x \in P$$

then

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k (\delta_t f)(kx) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \{f(kxt) - f(kx)\} =$$

$$= \varphi(xt) - \varphi(x) = \delta_t \varphi(x) \quad \text{for all} \quad x, t \in P.$$

Thus

$$\delta_t \varphi(x) = c(t),$$

i.e.

$$\varphi(xt) = \varphi(x) + c(t) \quad \text{for all} \quad x, t \in P. \tag{5}$$

Since $kx = x(ke)$ for $x \in P$ and k a positive integer, it follows that $\delta_{ke} f(x) = f(kx) - f(x)$, $x \in P$.

Thus

$$\begin{aligned} \varphi(x) &= \sum_{k=1}^{n} \binom{n}{k} (-1)^k f(kx) = \\ &= \sum_{k=1}^{n} \binom{n}{k} (-1)^k \{f(kx) - f(x)\} + \left\{ \sum_{k=1}^{n} \binom{n}{k} (-1)^k \right\} f(x) = \\ &= \sum_{k=1}^{n} \binom{n}{k} (-1)^k (\delta_{ke} f)(x) - f(x) \quad \text{for all} \quad x \in P. \end{aligned}$$

But $\Delta_y^n \delta_{ke} f(x) = 0$ for all $x, y \in P$ and every positive integer k. Hence, if we let $\tilde{g} = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \delta_{ke} f$ then $\Delta_y^n \tilde{g}(x) = 0$ for all $x, y \in P$ and

$$f = \tilde{g} - \varphi. \tag{6}$$

From (5) it follows that

$$\begin{aligned} \varphi(x) + c(s) + c(t) = \varphi(xs) + c(t) = \varphi(xst) = \\ = \varphi(x) + c(st) \end{aligned}$$

so that

$$c(st) = c(s) + c(t) \quad \text{for all } s, t \in P. \tag{7}$$

But, by (5), $\varphi(t) = \varphi(e) + c(t)$, $t \in P$. Hence by (6)

$$f(x) = \tilde{g}(x) - \varphi(e) - c(x) \quad \text{for all} \quad x \in P.$$

Thus (2) holds if we let $g(x) = \tilde{g}(x) - \varphi(e)$ and $h(x) = -c(x)$ for all $x \in P$. Since $\Delta_y^n \tilde{g}(x) = 0$ for all $x, y \in P$ and g differs from $\tilde{g}$ by a constant, (3) holds. Since $h = -c$ it follows from (7) that (4) holds as well. Thus (1) implies (2), (3) and (4).

Conversely, suppose (2), (3) and (4) hold. Then for every $x, t \in P$,

$$\begin{aligned} \delta_t f(x) &= \delta_t g(x) + \delta_t h(x) \\ &= g(xt) - g(x) + h(xt) - h(x) \\ &= g(xt) - g(x) + h(t) \end{aligned}$$

where we have used (2) and (4). Hence

$$\Delta_y^n \delta_t f(x) = \Delta_y^n \delta_t g(x) \quad \text{for all} \quad x, y, t \in P.$$

But

$$\Delta_y^n \delta_t g(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \{g((x+ky)t) - g(x+ky)\} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k g(xt+kyt) - \sum_{k=0}^{n} \binom{n}{k} (-1)^k g(x+ky) =$$

$$= \Delta_{yt}^n g(xt) - \Delta_y^n g(x) = 0 \quad \text{for all} \quad x, y, t \in P \quad \text{by} \quad (3).$$

Hence (1) holds and the proof is complete.

The case $n=2$ of theorem 3 generalizes theorem 1. To see this suppose P is the set of positive elements of an ordered field F and $\alpha: P \to G$. Notice that P admits division by 2. We claim that $\Delta_y^2 \alpha(x)=0$ for all $x, y \in P$ if and only if $2\alpha\left(\frac{x+y}{2}\right)=\alpha(x)+\alpha(y)$ for all $x, y \in P$.

Indeed, suppose $\Delta_y^2 \alpha(x)=0$ for all $x, y \in P$, i.e.

$$\alpha(x+2y)-2\alpha(x+y)+\alpha(x) = 0 \quad \text{for all} \quad x, y \in P.$$

Let $u, v \in P$ and assume without loss of generality that $u<v$. Then $v=u+p$ for some $p \in P$. But $p=2w$ for some $w \in P$ so $v=u+2w$. Hence $\alpha(u+2w)-2\alpha(u+w)+$ $+\alpha(u)=0$ or $2\alpha(u+w)=\alpha(u)+\alpha(v)$. But $u+w=u+\frac{v-u}{2}=\frac{u+v}{2}$ so

$$2\alpha\left(\frac{u+v}{2}\right) = \alpha(u)+\alpha(v).$$

Conversely, if $2\alpha\left(\frac{x+y}{2}\right)=\alpha(x)+\alpha(y)$ for all $x, y \in P$ then replacing x by $x+2y$ and y by x we find

$$\Delta_y^2 \alpha(x) = 0 \quad \text{for all} \quad x, y \in P.$$

Hence, if $f: P \to G$ then the following are equivalent for all $x, y, t \in P$:

(a) $2\delta_t f\left(\frac{x+y}{2}\right)=\delta_t f(x)+\delta_t f(y)$,

(b) $\Delta_y^2 \delta_t f(x)=0$,

(c) $f=\alpha+m$ where $\Delta_y^2 \alpha(x)=0$ and $m(xy)=m(x)+m(y)$,

(d) $f=\alpha+m$ where $2\alpha\left(\frac{x+y}{2}\right)=\alpha(x)+\alpha(y)$ and $m(xy)=m(x)+m(y)$.

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(Received September 29, 1977.)