Regular Power Semigroups

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Dedicated to the memory of Paul Rhodes

§ 1. Introduction

DUBREIL [1] appears to have been the first to investigate the semigroup of all subsets of a semigroup. In [4], LJAPIN states that the topic is worthy of further study and in [2], LAJOS has shown that if S is a regular semigroup and $P(S)$ denotes the semigroup of all non-empty subsets of S, then the set of all (1, 1)-ideals of S forms a regular subsemigroup of $P(S)$; WÖRZ—BUSEKROS has considered the regularity of further subsemigroups of $P(S)$ in [5]. In [3], TAMURA and SHAFER characterised those semigroups S for which $P(S)$ is a band, a chain or a lattice with 0, 1. In this paper we extend their result to a description of those semigroups S with identity for which $P(S)$ is regular.

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§ 2. The main result

Clearly, $P(S)$ is commutative if and only if S is also commutative, and if $P(S)$ is regular then S is regular since for each $a \in S$, $\{a\}=\{a\} \cdot B \cdot \{a\}$ for some $B \in P(S)$ and so there exists $x \in B$ with $a=axa$. The following result improves upon this fact.

Lemma. *If $P(S)$ is regular, then a^2 is idempotent for each $a \in S$.*

PROOF. Let $a \in S$. Then there exists $A \in P(S)$ such that $\{a, a^2\}=\{a, a^2\} \cdot A \cdot \{a, a^2\}$. Fix $x \in A$ and note that axa^2 equals a or a^2. In the first case, if also $axa=a$ then $a=a^2$, while if $axa=a^2$ then $a=a^3$. In the second case, if $a^2xa^2=a$ then $a=a^3$, and if $a^2xa^2=a^2$ then $a^2=a^3$. Hence in any case we obtain $(a^2)^2=a^2$, as required.

Theorem. *If S contains an identity 1 and $P(S)$ is regular, then $S=G \cup E$ where G is a group with at most 2 elements, E is a semigroup of primes, and if there exists $a \in G$, $a \neq 1$, then $au=ua=u$ for all $u \in E$. Conversely, if S is a semigroup of this form, then $P(S)$ is regular.*

PROOF We first show that for all $a \in S$, $a^2=1$ or $a^2=a$. To do this, consider $\{1, a\}$ and suppose $\{1, a\}=\{1, a\} \cdot B \cdot \{1, a\}$. Fix $x \in B$ and note that $ax \cdot 1$ equals

1 or a, and that $x \in \{1, a\}$. Hence if $ax=1$ then $a=1$ or $a^2=1$, while if $ax=a$ then $axa=1$ implies $a^2=1$ and $axa=a$ implies $a^2=a$.

Thus, $S=G \cup E$ where $G=\{a \in S: a^2=1\}$ and $E=\{a \in S: a^2=a \neq 1\}$. Suppose there exist $a, b \in G \setminus 1, a \neq b$, and consider $\{1, a, b\}$. If $\{1, a, b\}=\{1, a, b\} \cdot C \cdot \{1, a, b\}$, fix $x \in C$ and note that axb equals 1, a or b and that x also equals 1, a or b. So, in the first case, ab equals 1, b or a, a contradiction each time. If next $axb=a$ then $ax=1$ implies $b=a$, $ax=a$ implies $b=1$ and $ax=b$ implies $a=1$. Finally if $axb=b$ then $xb=1$ implies $a=b$, $xb=a$ implies $b=1$, and $xb=b$ implies $a=1$. Thus in all cases we obtain a contradiction and so $|G| \leqq 2$.

Now suppose there exists $a \in G \setminus 1$ and let $u \in E$ and consider $\{1, a, u\}$. If $\{1, a, u\}=\{1, a, u\} \cdot A \cdot \{1, a, u\}$, fix $x \in A$ and note that uxa equals 1, a or u and x also equals 1, a or u. It is easy to see that if uxa equals either 1 or a, we obtain a contradiction. Thus $uxa=u$. If now ux equals 1 or a, we have another contradiction. Hence $ux=u$ and finally $ua=u$. A dual argument shows that also $au=u$.

Finally let $u, v \in E$ and consider $\{1, u, v\}$. If $\{1, u, v\}=\{1, u, v\} \cdot A \cdot \{1, u, v\}$, fix $x \in A$ and note that uxv equals 1, u or v and x also equals 1, u or v. Thus, in the first case, we have $uv=1$ and so $v=1$, a contradiction. So, either $uxv=u$ (in which case we have $uv=u$) or $uxv=v$ (in which case $uv=v$).

For the converse we simply note that if S is any semigroup with the prescribed properties then $A=A^3$ for any non-empty $A \subseteq S$.

References

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