The common source of several inequalities concerning doubly stochastic matrices

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Summary

In this paper the author proves some inequalities concerning the permanents of matrices. One of them has an important role, because the others are consequences of this. One theorem of them is a generalization of the author's theorem proved in his paper [1]. The author coordinates a conjecture concerning the permanents of doubly stochastic matrices with every proved theorem. This conjectures are consequences of one of them too. The well-known conjecture of Van der Waerden is a special case of two of the above mentioned ones.

Key words: doubly stochastic matrices, permanent of a matrix, Bernstein's polynomial adjoined to a matrix, Van der Waerden conjecture.

1. Introduction

Let $\mathcal{M}$ denote the set of $n\times n$ matrices with real elements, where all row and column sums are 1. Let $\mathcal{H}\subset\mathcal{M}$ be the set of matrices with non-negative elements, i.e. the set of the so-called doubly stochastic matrices. Let $A_0\in\mathcal{H}$ be the matrix where all the entries are $1/n$. Let A^* denote the transpose of $A\in\mathcal{M}$. Denote Per A the permanent of $A\in\mathcal{M}$. The definition and the more important properties of the permanent can be found e.g. in [2].

Let $A=U\Lambda V^*$ be the polar representation of $A\in\mathcal{M}$, where U and V are orthogonal matrices and Λ is a diagonal matrix with non-negative elements. It is known ([1]) that the relation $A\in\mathcal{M}$ holds if and only if all elements of one of the columns of U and V, let this the first one, are equal to $1/\sqrt{n}$, and then the first diagonal element of Λ is equal to 1.

Let Γ_k $(k=1, \dots, n)$ be the set of the combinations of order k of the elements $1, \dots, n$ without repetition and without permutation, i.e. let

$$\Gamma_k=\{(i_1, \dots, i_k)|1\leqq i_1<\dots<i_k\leqq n\}.$$

By the help of the matrix $A=(a_{ij})\in\mathcal{M}$ we build the matrices

$$A_{i_1\dots i_k}^{j_1\dots j_k}=\begin{pmatrix} a_{i_1j_1} & \dots & a_{i_1j_k}\\ \cdot & \dots & \cdot \\ a_{i_kj_1} & \dots & a_{i_kj_k}\end{pmatrix} \tag{1.1}$$

$$(i_1, \dots, i_k)\in\Gamma_k, \quad (j_1, \dots, j_k)\in\Gamma_k.$$

Denote Sum $A_{i_1\dots i_k}^{j_1\dots j_k}$ the sum of the elements of the matrix (1, 1).

Let $A\in\mathcal{M}$. Now we introduce the following notations:

$$T_k(A)=\sum \operatorname{Per} A_{i_1\dots i_k}^{j_1\dots j_k},$$
$$R_k(A)=\sum \operatorname{Per} A_{i_1\dots i_k}^{j_1\dots j_k} \operatorname{Sum} A_{i_1\dots i_k}^{j_1\dots j_k},$$
$$S_k(A)=\frac{R_k(A)}{T_k(A)},\quad (k=1,\dots,n)$$
$$P_k(A)=\frac{T_{k+1}(A)}{T_k(A)},\quad (k=1,\dots,n-1),$$

where the summations are extended over all

$$(i_1,\dots,i_k)\in\Gamma_k,\quad (j_1,\dots,j_k)\in\Gamma_k.$$

Let $p\geqq 0$, $q\geqq 0$, $p+q=1$, Let $A\in\mathcal{M}$. Denote $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$ the positive semidefinite square root of AA^* and A^*A respectively.

Let us introduce the following notations:

$$T_k^{(p)}(A)=p^2T_k[(AA^*)^{1/2}]+q^2T_k[(A^*A)^{1/2}]+2pqT_k(A),$$
$$R_k^{(p)}(A)=p^2R_k[(AA^*)^{1/2}]+q^2R_k[(A^*A)^{1/2}]+2pqR_k(A)\quad (k=1,\dots,n).$$

Moreover let

$$S_k^{(p)}(A)=\frac{R_k^{(p)}(A)}{T_k^{(p)}A}\quad (k=1,\dots,n),$$
$$P_k^{(p)}(A)=\frac{T_{k+1}^{(p)}(A)}{T_k^{(p)}(A)}\quad (k=1,\dots,n-1).$$

We are proving the following theorems in this paper:

Theorem 1.1. *If $A\in\mathcal{M}$ and if $0\leqq p\leqq 1$, then*

$$S_k^{(p)}(A)\geqq\frac{k^2}{n}\quad (k=1,\dots,n)$$

with equality in the cases of $k=1,\dots,n-1$ if and only if $A=A_0$. In the case of $k=n$ equality holds for all $A\in\mathcal{M}$.

Theorem 1.2. *If $A\in\mathcal{M}$ and if $0\leqq p\leqq 1$, then*

$$P_k^{(p)}(A)\geqq\frac{(n-k)^2}{n(k+1)}\quad (k=1,\dots,n-1)$$

with equality if and only if $A=A_0$.

Theorem 1.3. *If $A\in\mathcal{M}$ and if $0\leqq p\leqq 1$, then*

$$T_k^{(p)}(A)\geqq\binom{n}{k}^2\frac{k!}{n^k}\quad (k=2,\dots,n),$$
$$R_k^{(p)}(A)\geqq\binom{n}{k}^2\frac{k!\cdot k^2}{n^{k+1}}\quad (k=1,\dots,n)$$

with equality if and only if $A=A_0$.

If $A\in\mathcal{M}$ and if $0\leqq x\leqq 1$, then obviously $A(x)=(1-x)A_0+xA\in\mathcal{M}$. Similarly if $A\in\mathcal{H}$, then $A(x)\in\mathcal{H}$.

The polynomial $B(x)=\operatorname{Per} A(x)$, $0\leqq x\leqq 1$ is the Bernstein's polynomial adjoined to the matrix A.

Let $A\in\mathcal{M}$, $0\leqq p\leqq 1$. Denote $B_p(x)$ the weighted mean of the Bernstein's polynomials adjoined to the matrices $(AA^*)^{1/2}, (A^*A)^{1/2}$, A with weights $p^2, q^2, 2pq$ respectively.

Theorem 1.4. *If $A\in\mathcal{M}$, $A\neq A_0$ and if $0\leqq p\leqq 1$, then $B_p'(x)\geqq 0$ when $0\leqq x\leqq 1$ with equality, if and only if $x=0$.*

In chapter 2 we prove the theorem 1.1., while in chapter 3 we show that theorems 1.2.—1.4. are consequences of 1.1. To prove these we use a lemma, which is proved in this chapter too. This lemma make connections among the quantities $T_k(A)$, $T_{k+1}(A)$, and $R_k(A)$.

We deal with the following conjectures in chapter 4:

Conjecture 1.1. If $A\in\mathcal{H}$, then

$$S_k(A)\geqq\frac{k^2}{n}\quad(k=1,\dots,n)$$

with equality in the cases of $k=1,\dots,n-1$ if and only if $A=A_0$. In the case of $k=n$ equality holds for all $A\in\mathcal{H}$.

Conjecture 1.2. If $A\in\mathcal{H}$, then

$$(k+1)P_k(A)\geqq\frac{(n-k)^2}{n}\quad(k=1,\dots,n-1)$$

with equality if and only if $A=A_0$.

Conjecture 1.3. If $A\in\mathcal{H}$, then

$$T_k(A)\geqq\binom{n}{k}^2\frac{k!}{n^k}\quad(k=1,\dots,n)$$

with equality in the cases of $k=2,\dots,n$ if and only if $A=A_0$. In the case of $k=1$ equality holds for all $A\in\mathcal{H}$.

Conjecture 1.4. If $A\in\mathcal{H}$, then

$$R_k(A)\geqq\binom{n}{k}^2\frac{k!\,k^2}{n^{k+1}}\quad(k=1,\dots,n)$$

with equality if and only if $A=A_0$.

Conjecture 1.5. If $A\in\mathcal{H}$, $A\neq A_0$, then the Bernstein's polynomial $B(x)$ adjoined to A is strictly monoton increasing function on the interval $0<x\leqq 1$ and $B'(0)=0$.

From conjectures 1.3. and 1.5. we get as a special case the following well-known conjecture of Van der Waerden:

If $A\in\mathcal{H}$, then $\operatorname{Per} A\geqq n!/n^n$ with equality if and only if $A=A_0$.

In chapter 4 we show too that the conjectures 1.2.—1.5. are consequences of 1.1. If $A \in \mathcal{H}$ is a symmetric positive semidefinite matrix, then the validity of the conjectures 1.1.—1.5. are consequences of theorems 1.1.—1.4.

We successeded to prove the conjecture 1.1. generally only in the case of $k=1, 2$, and also in the case of $k=1$ for $A \in \mathcal{M}$. If $k=1$ we can give a certain positive neighbourhood of the origin in a way that for x's chosen from it, $A(x)$ satisfies the conjecture of Van der Waerden (corollary 4.1.).

2. The proof of theorem 1.1.

In this chapter theorem 1.1. will be proved first in the following form:

Theorem 2.1. *If* $A \in \mathcal{M}$ *and if* $0 \leqq p \leqq 1$, *then*

$$R_k^{(p)}(A) \geqq \frac{k^2}{n} T_k^{(p)}(A) \quad (k = 1, \ldots, n)$$

with equality in the cases of $k=1, \ldots, n-1$ *if and only if* $A=A_0$. *In the case of* $k=n$ *equality holds for all* $A \in \mathcal{M}$.

Proof. The last statement of the theorem is evident, since $\operatorname{Sum} A = n$ for all $A \in \mathcal{M}$. Thus it is sufficient to restrict our attention only to the cases of $k=1, \ldots, \ldots, n-1$.

Let

$$A = (a_{ik}) = U \Lambda V^*$$

be the polar representation of A, where U and V are orthogonal matrices and Λ is a diagonal matrix with non-negative elements. If

$$U = (u_{ij}), \quad V = (v_{ij}), \quad \Lambda = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix},$$

then

$$a_{ij} = \sum_{s=1}^{n} u_{is} v_{js} \lambda_s,$$

consequently

$$\operatorname{Sum} A_{i_1 \ldots i_k}^{j_1 \ldots j_k} = \sum_{s=1}^{n} \lambda_s x_{i_1 \ldots i_k}^{(s)} y_{j_1 \ldots j_k}^{(s)},$$

where

$$x_{i_1 \ldots i_k}^{(s)} = \sum_{\alpha=1}^{k} u_{i_\alpha s}, \quad y_{i_1 \ldots i_k}^{(s)} = \sum_{\alpha=1}^{k} v_{i_\alpha s}.$$

Since $A \in \mathcal{M}$, one of its eigenvalues is equal to 1. Let $\lambda_1 = 1$. In this case

$$u_{k1} = v_{k1} = 1/\sqrt{n} \quad (k = 1, \ldots, n).$$

Therefore

$$\operatorname{Sum} A_{i_1 \ldots i_k}^{j_1 \ldots j_k} = \frac{k^2}{n} + \sum_{s=2}^{n} \lambda_s x_{i_1 \ldots i_k}^{(s)} y_{j_1 \ldots j_k}^{(s)},$$

thus

$$R_k(A)-\frac{k^2}{n}T_k(A)=\sum \operatorname{Per} A^{j_1\dots j_k}_{i_1\dots i_k}\sum_{s=2}^{n}\lambda_s x^{(s)}_{i_1\dots i_k} y^{(s)}_{j_1\dots j_k}=\sum_{s=2}^{n}\lambda_s Q_s(A), \tag{2.1}$$

where

$$Q_s(A)=\sum \operatorname{Per} A^{j_1\dots j_k}_{i_1\dots i_k} x^{(s)}_{i_1\dots i_k} y^{(s)}_{j_1\dots j_k}.$$

Here and in (2. 1) the combinations $i_1, \dots, i_k$ and $j_1, \dots, j_k$ run independently from one another over Γ_k. Let $U_{i_1\dots i_k}$ and $V_{i_1\dots i_k}$ denote the $k\times n$ matrices, which consist of rows of U and V with indices $i_1, \dots, i_k$ respectively. In this case

$$A^{j_1\dots j_k}_{i_1\dots i_k}=U_{i_1\dots i_k}\Lambda V^*_{j_1\dots j_k}.$$

Using the Cauchy—Binet expansion formula ([2]) for the last identity, we get

$$\operatorname{Per} A^{j_1\dots j_k}_{i_1\dots i_k}=\sum\frac{\lambda_1^{\beta_1}\dots\lambda_n^{\beta_n}}{\beta_1!\dots\beta_n!}\operatorname{Per} C_{\beta_1\dots\beta_n}(U_{i_1\dots i_k})\cdot\operatorname{Per} C_{\beta_1\dots\beta_n}(V_{j_1\dots j_k}), \tag{2.2}$$

where the summation is extended over all non-negative integers $\beta_1, \dots, \beta_n$ satisfying the equality $\beta_1+\dots+\beta_n=k$. $C_{\beta_1\dots\beta_n}(U_{i_1\dots i_k})$ denotes the $k\times k$ matrix, which contains certain columns of matrix $U_{i_1\dots i_k}$. Namely the j-th column of $U_{i_1\dots i_k}$ appears β_j-times $(j=1, \dots, n)$ in $C_{\beta_1\dots\beta_n}(U_{i_1\dots i_k})$ and $\beta_1+\dots+\beta_n=k$.

Using the obtained representation (2. 2) of $\operatorname{Per} A^{j_1\dots j_k}_{i_1\dots i_k}$, we get

(2.3)

$$Q_s(A)=\sum\frac{\lambda_1^{\beta_1}\dots\lambda_n^{\beta_n}}{\beta_1!\dots\beta_n!}\Big(\sum\operatorname{Per} C_{\beta_1\dots\beta_n}(U_{i_1\dots i_k})x^{(s)}_{i_1\dots i_k}\Big)\Big(\sum\operatorname{Per} C_{\beta_1\dots\beta_n}(V_{i_1\dots i_k})y^{(s)}_{i_1\dots i_k}\Big),$$

where the first summation is extended over all non-negative integers $\beta_1, \dots, \beta_n$ satisfying the equality $\beta_1+\dots+\beta_n=k$, while in the second and in the third ones $i_1, \dots, i_k$ runs over Γ_k.

Let now $s\geqq 2$ and $\beta_1=k$. In this case of

$$\sum_{(i_1,\dots,i_k)\in\Gamma_k}\operatorname{Per} C_{\beta_1=k}(U_{i_1\dots i_k})x^{(s)}_{i_1\dots i_k}=k!\left(\frac{1}{\sqrt{n}}\right)^k\binom{n-1}{k-1}(u_{1s}+\dots+u_{ns})=0,$$

because in the case of $s\geqq 2$ the vector $(u_{1s}, \dots, u_{ns})$ is orthogonal to the n-dimensional vector with the common components $1/\sqrt{n}$.

If $\beta_s=1$ $(2\leqq s\leqq n)$, $\beta_1=k-1$, then

$$\operatorname{Per} C_{\beta_1=k-1,\beta_s=1}(U_{i_1\dots i_k})=(k-1)!\left(\frac{1}{\sqrt{n}}\right)^{k-1}(u_{i_1 s}+\dots+u_{i_k s}),$$

consequently

$$\operatorname{Per} C_{\beta_1=k-1,\beta_s=1}(U_{i_1\dots i_k})x^{(s)}_{i_1\dots i_k}=(k-1)!\left(\frac{1}{\sqrt{n}}\right)^{k-1}(u_{i_1 s}+\dots+u_{i_k s})^2.$$

Thus

(2.4)

$$Q_s(A)=$$

$$=[(k-1)!]^2\left(\frac{1}{n}\right)^{k-1}\lambda_s\sum_{(i_1,\dots,i_k)\in\Gamma_k}(u_{i_1 s}+\dots+u_{i_k s})^2\sum_{(i_1,\dots,i_k)\in\Gamma_k}(v_{i_1 s}+\dots+v_{i_k s})^2+\varrho_s(A),$$

where we can get $\varrho_s(A)$ from the representation (2. 3) of $Q_s(A)$, if the first summation is extended over all non-negative integers $\beta_1, \ldots, \beta_n$ satisfying the conditions $\beta_1+\ldots+\beta_n=k$, $\beta_1 \leqq k-2$.

Our statement is

$$\sum_{(i_1,\ldots,i_k)\in\Gamma_k} (u_{i_1 s}+\ldots+u_{i_k s})^2 > 0. \tag{2.5}$$

Otherwise

$$u_{i_1 s}+\ldots+u_{i_k s} = 0$$

for all $(i_1, \ldots, i_k)\in\Gamma_k$, therefore $u_{is}=0$ for $i=1, \ldots, n$, because $1\leqq k\leqq n-1$, contradicting to the orthogonality of U.

We turn now to the proof of the theorem 2.1.

On the basis of formula (2.1) we get

$$R_k^{(p)}(A)-\frac{k^2}{n}T_k^{(p)}(A) = \sum_{s=2}^{n} \lambda_s Q_s^{(p)}(A), \tag{2.6}$$

where

$$Q_s^{(p)}(A) = p^2 Q_s[(AA^*)^{1/2}]+q^2 Q_s[(A^*A)^{1/2}]+2pqQ_s(A).$$

Taking (2, 4) into account

$$\begin{aligned} Q_s^{(p)}(A) &= [(k-1)!]^2\left(\frac{1}{n}\right)^{k-1}\cdot\lambda_s\cdot \\ &\cdot\Big(p\sum_{(i_1,\ldots,i_k)\in\Gamma_k}(u_{i_1 s}+\ldots+u_{i_k s})^2+q\sum_{(i_1,\ldots,i_k)\in\Gamma_k}(v_{i_1 s}+\ldots+v_{i_k s})^2\Big)^2+ \\ &+\sum\frac{\lambda_1^{\beta_1}\ldots\lambda_n^{\beta_n}}{\beta_1!\ldots\beta_n!}\cdot\Big(p\sum_{(i_1,\ldots,i_k)\in\Gamma_k}\operatorname{Per} C_{\beta_1\ldots\beta_n}(U_{i_1\ldots i_k})\cdot x^{(s)}_{i_1\ldots i_k}+ \\ &+q\sum_{(i_1,\ldots,i_k)\in\Gamma_k}\operatorname{Per} C_{\beta_1\ldots\beta_n}(V_{i_1\ldots i_k})\,y^{(s)}_{i_1\ldots i_k}\Big)^2, \end{aligned} \tag{2.7}$$

where the first summation of the second term is extended over the non-negative integers $\beta_1, \ldots, \beta_n$ satisfying the conditions $\beta_1+\ldots+\beta_n=k$, $\beta_1\leqq k-2$.

On the basis of (2, 5) the expression in the breckets of the first term of (2.7) is positive for all $0\leqq p\leqq 1$, therefore the first term is zero if and only if $\lambda_s=0$. The second term is non-negative and equal to zero if $\lambda_s=0$ for $s=2, \ldots, n$. This complete the proof of theorem 2.1. on the basis of (2.6).

If we want to prove theorem 1.1., it is necessary to show also that $T_k^{(h)}(A)\neq 0$. On the basis of (2, 2)

$$T_k^{(p)}(A) = \sum\sum\frac{\lambda_1^{\beta_1}\ldots\lambda_n^{\beta_n}}{\beta_1!\ldots\beta_n!}\cdot\big(p\operatorname{Per} C_{\beta_1\ldots\beta_n}(U_{i_1\ldots i_k})+q\operatorname{Per} C_{\beta_1\ldots\beta_n}(V_{j_1\ldots j_k})\big)^2,$$

where the first summation is extended over all

$$(i_1, \ldots, i_k)\in\Gamma_k, \quad (j_1, \ldots, j_k)\in\Gamma_k,$$

while the second one is extended over the non-negative integers $\beta_1, \ldots, \beta_n$ satisfying the condition $\beta_1+\ldots+\beta_n=k$. Now we retain only the member for $\beta_1=n$. In this case $\lambda_1=1$ and all elements of $U_{i_1\ldots i_k}$ and $V_{j_1\ldots j_k}$ are $1/\sqrt{n}$. Thus

$$T_k^{(p)}(A) \geqq \binom{n}{k}^2 \frac{k!}{n^k},$$

i.e. the proof of theorem 1.1. is completed.

3. The proof of theorems 1.2—1.4.

a) In this section we show that theorems 1.2.—1.4. are consequences of the 1.1. First we prove a lemma, which has a fundamental role in the proofs of this chapter.

Lemma 3.1. *If $A\in\mathcal{M}$, then*

$$T_{k+1}(A) = \frac{1}{k+1}\left[(n-2k)T_k(A)+R_k(A)\right] \quad (k=1, \ldots, n-1).$$

Proof. Let $A=(a_{ij})$. Denote $\mathcal{V}_k$ the set of the variations of order k of the elements $1, \ldots, n$ without repetition. Then

$$T_k(A) = \frac{1}{k!}\sum \operatorname{Per} A_{i_1\ldots i_k}^{j_1\ldots j_k}, \tag{3.1}$$

where in the summation $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ run independently from one another over $\mathcal{V}_k$.

Let us write $T_{k+1}(A)$ in a similar form. In this form the sum of the term containing the product $a_{11}\ldots a_{kk}$ is equal to

$$\sigma = a_{11}\ldots a_{kk}[(a_{k+1\,k+1}+\ldots+a_{n\,k+1})+\ldots+(a_{k+1\,n}+\ldots+a_{nn})].$$

Taking $A\in\mathcal{M}$ into account,

$$\sigma = a_{11}\ldots a_{kk}\{[1-(a_{1k+1}+\ldots+a_{kk+1})]+\ldots+[1-(a_{1n}+\ldots+a_{kn})]\} =$$
$$= a_{11}\ldots a_{kk}\{n-k-[(a_{1k+1}+\ldots+a_{1n})+\ldots+(a_{kk+1}+\ldots+a_{kn})]\}.$$

If we use again that $A\in\mathcal{M}$, then we get

$$\sigma = a_{11}\ldots a_{kk}\{n-k-[1-(a_{11}+\ldots+a_{1k})]-\ldots-[1-(a_{k1}+\ldots+a_{kk})]\} =$$
$$= a_{11}\ldots a_{kk}(n-2k+\operatorname{Sum} A_{1\ldots k}^{1\ldots k}).$$

Thus in $T_{k+1}(A)$ written in the form under (3.1) the sum of the terms containing the factor $\operatorname{Per} A_{1\ldots k}^{1\ldots k}$ is equal to

$$\operatorname{Per} A_{1\ldots k}^{1\ldots k}\{n-2k+\operatorname{Sum} A_{1\ldots k}^{1\ldots k}\}.$$

Similarly we get that in $T_{k+1}(A)$ written in the form under (3.1), the sum of the terms containing the factor $\operatorname{Per} A_{i_1\ldots i_k}^{j_1\ldots j_k}$ is equal to

$$\operatorname{Per} A_{i_1\ldots i_k}^{j_1\ldots j_k}\{n-2k+\operatorname{Sum} A_{i_1\ldots i_k}^{j_1\ldots j_k}\}.$$

Therefore

$$T_{k+1}(A) = \frac{1}{(k+1)!} \sum \text{Per } A_{i_1 \dots i_k}^{j_1 \dots j_k} (n-2k+\text{Sum } A_{i_1 \dots i_k}^{j_1 \dots j_k}),$$

where in the summation $i_1, \dots, i_k$ and $j_1, \dots, j_k$ run independently from one another over $\mathscr{V}_k$. The last formula contains the statement of the lemma 3.1.

It is easy to show the following corollary of the lemma 3.1.:

Corollary 3.1. *If* $A \in \mathscr{M}$ *and if* $0 \leqq p \leqq 1$, *then*

$$T_{k+1}^{(p)}(A) = \frac{1}{k+1} [(n-2k) T_k^{(p)}(A) + R_k^{(p)}(A)]$$

$$(k = 1, \dots, n-1).$$

Since $T_k(A) > 0$ for $A \in \mathscr{H}$, therefore

$$(k+1) P_k(A) = n-2k+S_k(A),$$

and similarly for $A \in \mathscr{M}$

$$(k+1) P_k^{(p)}(A) = n-2k+S_k^{(p)}(A)$$

if $k=1, \dots, n-1$. Thus

$$(k+1) P_k(A) + (n-k+1) P_{n-k}(A) = S_k(A) + S_{n-k}(A)$$

for $A \in \mathscr{H}$ and

$$(k+1) P_k^{(p)}(A) + (n-k+1) P_{n-k}^{(p)}(A) = S_k^{(p)}(A) + S_{n-k}^{(p)}(A)$$

for $A \in \mathscr{M}$ and for $k=1, \dots, n-1$. From here we get the following result:

Corollary 3.2. *The identities*

$$\sum_{k=1}^{n-1} (k+1) P_k(A) = \sum_{k=1}^{n-1} S_k(A),$$

and

$$\sum_{k=1}^{n-1} (k+1) P_k^{(p)}(A) = \sum_{k=1}^{n-1} S_k^{(k)}(A)$$

hold for $A \in \mathscr{H}$ *and for* $A \in \mathscr{M}$ *respectively.*

b) We turn now to the proof of theorems 1.2.—1.4.

On the basis of corollary 3.1. and theorem 1.1. we get the following result:

Theorem 3.1. *If* $A \in \mathscr{M}$, *then theorems 1.1. and 1.2. are equivalent.*

From this theorem we obtain the proof of theorem 1.2.

The first statement of the theorem 1.3. follows from theorem 1.2. on the basis of the identity

$$T_k^{(p)}(A) = T_1(A) \prod_{i=1}^{k-1} P_i^{(p)}(A),$$

while we get the second statement of the mentioned theorem from here and from the theorem 1.1. on the basis of

$$R_k^{(p)}(A) = S_k^{(p)}(A)\,T_k^{(p)}(A).$$

As a consequence of theorems 1.1.—1.3. we obtain the following corollary:

Corollary 3.3. *If $A \in \mathcal{M}$ is a symmetric positive semidefinite matrix, then*

$$S_k(A) \geqq \frac{k^2}{n} \quad (k = 1, \dots, n-1),$$

$$P_k(A) \geqq \frac{(n-k)^2}{n(k+1)} \quad (k = 1, \dots, n-1), \quad T_k(A) \geqq \binom{n}{k}^2 \frac{k!}{n^k} \quad (k = 2, \dots, n),$$

$$R_k(A) \geqq \binom{n}{k}^2 \frac{k!\,k^2}{n^{k+1}} \quad (k = 1, \dots, n)$$

with equality if and only if $A = A_0$.

We shall prove now the theorem 1.4.

Let $B(x)$ be the Bernstein's polynomial adjoined to $A \in \mathcal{M}$. By elementary permanent transformation we get that

$$B(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} c(k),$$

where

$$c(0) = \frac{n!}{n^n},$$

$$c(k) = \frac{1}{\binom{n}{k}} \sum \operatorname{Per} C_{i_1 \dots i_k}(A).$$

The summation is extended for all $i_1, \dots, i_k$ over Γ_k. Here $C_{i_1 \dots i_k}(A)$ is the $n \times n$ matrix the columns of which with indices $i_1, \dots, i_k$ are equal to the columns of A with same indices, and its other elements are equal to $1/n$. It is not difficult to verify that

$$c(k) = \frac{1}{\binom{n}{k}} \cdot \frac{(n-k)!}{n^{n-k}} T_k(A) \quad (k = 1, \dots, n).$$

Let

$$\Delta^0 c(k) = c(k) \quad (k = 0, 1, \dots, n),$$

$$\Delta^\nu c(k) = \Delta^{\nu-1} c(k+1) - \Delta^{\nu-1} c(k)$$

$$(k = 0, 1, \dots, n-\nu;\ \nu = 1, \dots, n).$$

Since

$$c(0) = c(1) = \frac{n!}{n^n},$$

therefore ([3], 179) we obtain

$$B'(x) = n\sum_{k=1}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-k}\Delta c(k), \tag{3.2}$$

and consequently

$$B^{(\nu)}(x) = n(n-1)\dots(n-\nu+1)\sum_{k=0}^{n-\nu}\binom{n-\nu}{k}x^k(1-x)^{n-k}\Delta^\nu c(k).$$

If we use this, the formula

$$B(x) = \sum_{\nu=0}^{n}\frac{x^\nu}{\nu!}B^{(\nu)}(0) = \frac{n!}{n^n}+\sum_{\nu=2}^{n}\binom{m}{\nu}\Delta^\nu c(0)x^\nu \tag{3.3}$$

holds.

Similarly we can find that

$$B^{(\nu)}(x) = n(n-1)\dots(n-\nu+1)\sum_{i=0}^{n-\nu}\frac{x^i}{i!}B^{(\nu+i)}(0) =$$

$$= n(n-1)\dots(n-\nu+1)\sum_{i=0}^{n-\nu}\binom{n-\nu}{i}\Delta^{\nu+i}c(0)\cdot x^i \quad (\nu = 1, \dots, n).$$

We turn now to the proof of the theorem 1.4.

Taking (3.2) into account, we get

$$B_p'(x) = n\sum_{k=1}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-k}\Delta c_p(k),$$

where

$$c_p(0) = \frac{n!}{n^n}, \quad c_p(k) = \frac{1}{\binom{n}{k}}\cdot\frac{(n-k)!}{n^{n-k}}T_k^{(p)}(A)$$

$$(k = 1, \dots, n).$$

On the basis of theorem 1.2. we get that

$$\frac{c_p(k+1)}{c_p(k)} = \frac{n(k+1)}{(n-k)^2}P_k^{(p)}(A) > 1,$$

because $A \neq A_0$, i.e.

$$\Delta c_p(k) > 0 \quad (k = 1 \dots, n-1),$$

which gives the statement of the theorem.

Corollary 3.4. *If $A \in \mathcal{M}$ is a symmetric positive semidefinite matrix, and $A \neq A_0$, then the Bernstein's polynomial $B(x)$ adjoined to A is strictly monoton increasing on the interval $0 < x \leqq 1$, and $B'(0) = 0$.*

On the basis of (3.2) we obtain

$$B(x) = \frac{n!}{n^n}+\sum_{k=1}^{n-1}B(k+1, n-k; x)\Delta c(k), \quad 0 \leqq x \leqq 1$$

as an another representation for the Bernstein's polynomial adjoined to A, where $B(k+1, n-k; x)$ is the distribution function of the Beta distribution with parameters $k+1, n-k$ (see e.g. [4], 213).

4. On the conjectures

a) In this section we show that the conjectures 1.1. and 1.2. are equivalent, and that the conjectures 1.3.—1.5. are consequences of 1.1.

If $A\in\mathscr{H}$, then it is easily to see that $T_k(A)>0$ for $k=1, \dots, n$.

Applying lemma 3.1. we get the following result:

Theorem 4.1. *If $A\in\mathscr{H}$, then the conjectures* 1.1. *and* 1.2. *are equivalent.*

We say that the matrix $A\in\mathscr{H}$ satisfies the condition C_k, if $S_k(A)\geqq k^2/n$ holds with equality if and only if $A=A_0$. Here k is a positive integer satisfying the condition $k\leqq n-1$. If $k=n$, condition C_n is satisfied automatically, since $\text{Sum}\, A=n$ for $A\in\mathscr{H}$.

Theorem 4.2. *If $A\in\mathscr{H}$ satisfies conditions C_j for $j=1, \dots, k-1$ $(k=2, \dots, n)$, then*

$$T_j(A) \geqq \binom{n}{j}^2 \frac{j!}{n^j} \quad (j = 1, \dots, k)$$

with equality in the cases of $j=2, \dots, k$ if and only if $A=A_0$. If $j=1$, *then equality holds for all $A\in\mathscr{H}$.*

Proof. If we start from $T_1(A)=n$, and then we apply the condition of the theorem consecutively in the lemma 3.1., we get the statement of the theorem.

Theorem 4.3. *If $A\in\mathscr{H}$ satisfies the condition C_j for $j=1, \dots, k$ $(k=1, \dots, n)$,* then

$$R_j(A) \geqq \binom{n}{j}^2 \frac{j!\, j^2}{n^{j+1}} \quad (j = 1, \dots, k)$$

with equality if and only if $A=A_0$.

Proof. We obtain the statement from theorem 4.2. using the condition of the theorem.

Theorem 4.4. *If the matrix $A\in\mathscr{H}$, $A\neq A_0$ satisfies the condition C_k for $k=1, \dots, n$, then the Bernstein's polynomial $B(x)$ adjoined to A is strictly monoton increasing on the interval $0<x\leqq 1$, and $B'(0)=0$.*

Proof. This theorem can be similarly proved on the basis of (3.5) using the condition of the theorem as we proved one 1.4.

b) It follows from the corollaries 3.3. and 3.4. that the conjectures 1.1.—1.5. are true if $A\in\mathscr{H}$ is a symmetric positive semidefinite matrix. We had no success in the proof of the conjectures 1.1.—1.5. for arbitrary $A\in\mathscr{H}$, only in the cases of $k=1, 2$.

Let $A=(a_{ij})$ be a $n\times n$ matrix with real elements, and let

$$h_\nu(A) = \sum_{i,j=1}^{n} a_{ij}^{\nu} \quad (\nu = 1, 2, \dots).$$

If $A\in\mathcal{M}$, then $h_1(A)=n$ and it can be easily shown that $h_2(A)\geqq 1$, and that

$$h_\nu(A)\geqq\left(\frac{1}{n}\right)^{\nu-2}\quad(\nu=3,4,\ldots)$$

for $A\in\mathcal{H}$. In the last two inequality equality is if and only if $A=A_0$.

Theorem 4.5. If $A\in\mathcal{M}$, then

$$S_1(A)\geqq\frac{1}{n}\quad(n=2,3,\ldots),$$

with equality if and only if $A=A_0$.

Proof. In this case

$$R_1(A)=T_1(A)=h_2(A)-1,$$

thus the statement is trivial according to the above mentioned remarks.

It can not be hoped that we can extend this theorem for $k>1$ provided $A\in\mathcal{M}$. Namely let e.g. the matrix

$$A=\begin{pmatrix}-4 & 4 & 1\\ 3 & -4 & 2\\ 2 & 1 & -2\end{pmatrix}\in\mathcal{M}$$

be given. Since in this case

$$\operatorname{Per}A=-53,\quad T_2(A)=27,\quad R_2(A)=-56,$$

thus condition C_2 is not satisfied.

Theorem 4.6. *If $A\in\mathcal{H}$ then conjecture* 1.1. *is satisfied for* $k=2$.

Proof. It is natural in this case that $n\geqq 2$ and if $n=2$ the statement is evident. Thus we investigate only the case on $n>2$.

Let $A=(a_{ij})\in\mathcal{H}$. Now let us disjoin the sums

$$R_2(A)=\sum\operatorname{Per}\begin{pmatrix}a_{ik}\,a_{il}\\ a_{jk}\,a_{jl}\end{pmatrix}(a_{ik}+a_{il}+a_{jk}+a_{jl});$$

$$T_2(A)=\sum\operatorname{Per}\begin{pmatrix}a_{ik}\,a_{il}\\ a_{jk}\,a_{jl}\end{pmatrix}$$

— where in the summation i, j and k, l run independently from one another over Γ_2 — on the basis of the identity

$$\sum_{\substack{(i,j)\in\Gamma_2\\ (k,l)\in\Gamma_2}}=\frac{1}{2}\sum_{i\neq j,\,k\neq l}=\frac{1}{2}\Big\{\sum_{i,j,k,l}-\sum_{i=j,k,l}-\sum_{i,j,k=l}+\sum_{i=j,k=l}\Big\}.$$

Then we taken the following relations into consideration:

$$\sum_{i,k,l}a_{ik}^2a_{il}=h_2(A),\quad \sum_{i,k,j,l}a_{ik}^2a_{jl}=nh_2(A),$$

$$\sum_{i,k}a_{ik}^3=h_3(A),\quad \sum_{i,j,k,l}a_{ik}a_{jl}a_{il}=n,$$

$$\sum_{i,j,k,l}a_{ik}a_{jl}=n^2,\quad \sum_{i,k,l}a_{ik}a_{il}=n,$$

and we get that

$$R_2(A) = 2nh_2(A)+2n-8h_2(A)+4h_3(A),$$
$$T_2(A) = n^2-2n+h_2(A).$$

Consequently

$$R_2(A)-\frac{4}{n}T_2(A) = 4\big(h_3(A)-\frac{1}{n}h_2(A)\big)+2(n-4)\big(h_2(A)-1\big). \tag{4.1}$$

If β_k $(k=1, 2, \ldots)$ denotes the k-th absolute moment of a discret finite random variable, the well-known inequality $\beta_{k+1}\beta_{k-1}\geqq\beta_k^2$ holds ([4], 103). Applying this in our case if $k=2$, we get that

$$h_3(A) \geqq \frac{1}{n}h_2^2(A).$$

Using this in (4.1) we obtain

$$R_2(A)-\frac{4}{n}T_2(A) \geqq 2\big(n-4+\frac{2}{n}h_2(A)\big)\big(h_2(A)-1\big).$$

Thus the proof of the theorem is completed for $n\geqq 4$. It only remains for us to investigate the case of $n=3$. Let us use (4.1) for $n=3$, then

$$R_2(A)-\frac{4}{9}T_2(A) = \frac{2}{3}[6h_3(A)-5h_2(A)+3].$$

This expression is non-negative if and only if

$$h_3(A) \geqq \frac{5}{6}h_2(A)-\frac{1}{2} \geqq \frac{1}{3}.$$

This inequality is satisfied by all $A\in\mathscr{H}$ with equality if and only if $A=A_0$.

Applying the two last theorems, we get the following results using theorems 4.1.—4.3.:

Theorem 4.7. *If $A\in\mathscr{M}$ and if $n\geqq 2$, then*

$$T_2(A) \geqq \frac{1}{3}(n-1)^2, \quad R_1(A) \geqq 1,$$

with equality if and only if $A=A_0$.

Theorem 4.8. *If $A\in\mathscr{H}$ and if $n\geqq 3$, then*

$$T_3(A) \geqq \frac{1}{6n}(n-1)^2(n-2)^2,$$

$$R_2(A) \geqq \frac{2}{n}(n-1)^2,$$

$$P_2(A) \geqq \frac{(n-2)^2}{3n}$$

with equality if and only if $A=A_0$.

On the basis of lemma 3.2. and in consequence of theorem 4.5. we get

$$\Delta^2 c(0) = \Delta c(1) > 0.$$

Thus using the formula (3.3) we obtain the following theorem:

Theorem 4.9. *If* $A \in \mathcal{M}$, *then we can find a* $\delta > 0$, *so that*

$$\text{Per}((1-x)A_0 + xA) \geqq \frac{n!}{n^n}, \tag{4.2}$$

provided $0 < x \leqq \delta$, *with equality if and only if* $A = A_0$.

Corollary 4.1. *If* $A \in \mathcal{H}$, *then the matrices* (4.2) *satisfy the conjecture of Van der Waerden.*

On the basis of these it is easy to show that the conjecture of Van der Waerden is equivalent to the following one:

Conjecture 4.1. The only solution of the matrix equation

$$\text{Per}\, A = \frac{n!}{n^n}, \quad A \in \mathcal{H},$$

is $A = A_0$.

Namely if $\text{Per}\, A < \frac{n!}{n^n}$ for a matrix $A \in \mathcal{H}$, then there exists a $0 < x_0 < 1$ on the basis of identity (3.3) and of corollary 4.1. such, that

$$\text{Per}((1-x_0)A_0 + x_0 A) = \frac{n!}{n^n}$$

and $(1-x_0)A_0 + x_0 A \neq A_0$ contradictinction to the conjecture 4.1.

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