Neatness in certain algebraic lattices

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Dedicated to Professor K. Iséki on his 60th birthday

1. Introduction

It was proved independently by T. J. HEAD [4, Theorem 4] and A. KERTÉSZ [6, Satz 2] (s. also A. KERTÉSZ [5, Theorem 2]) that in a modular algebraic lattice L the property "L is atomistic" is equivalent to the property "every element of L is pure in L". A similar result appears in a slightly different context in J. E. DELANY [2, Corollary 4].

Combining some results of MAEDA—MAEDA [9] and of CRAWLEY—DILWORTH [1] we sharpen the theorems of Head, Kertész and Delany by replacing modularity by certain weaker covering conditions. As a by-product we get a partial answer to a question posed by A. Kertész.

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2. Basic Concepts

For two elements x, y of a lattice L we write $x \prec y$ if $x < y$ and if $x \leqq a \leqq y$ $(a \in L)$ implies either $a = x$ or $a = y$.

In a lattice with 0 an element p is called an atom if $0 \prec p$. Dually, in a lattice with 1 an element m is called a dual atom if $m \prec 1$.

A lattice with 0 is called atomistic, if each of its elements $(\neq 0)$ is the join of the atoms contained in this element.

KERTÉSZ [5, 6] and HEAD [4] extended the notion of a pure subgroup of an abelian group to algebraic lattices in the following way:

Definition. Let L be an algebraic lattice. An element $b \in L$ is called pure in L if b has for every compact element $s \in L$ a relative complement in the interval $[0, b \vee s]$.

Both Kertész and Head found that many results on pure subgroups of abelian groups hold in a more general setting involving pure elements of algebraic modular lattices. In the theory of abelian groups the notion of a pure subgroup is weakened by the concept of neat subgroup (s.e.g. FUCHS [3]). DELANY [2] introduced the lattice theoretic counterpart of a neat subgroup as follows:

Definition. An element n of a lattice L with 0 is called neat in L if $n \prec n'$ $(n' \in L)$ implies the existence of an element $a \in L$ such that $a \vee n = n'$ and $a \wedge n = 0$.

It is obvious that in an algebraic lattice a pure element is a fortiori a neat element (but not conversely). Delany [2] investigated some properties of neat subgroups of abelian groups which can be extended to algebraic modular lattices or even more general lattices.

We shall deal here with pure and with neat elements in algebraic lattices which satisfy certain generalizations of modularity. These generalizations of modularity will be given by means of certain covering conditions.

According to a classical result of Dedekind, a modular lattice satisfies the neighborhood condition

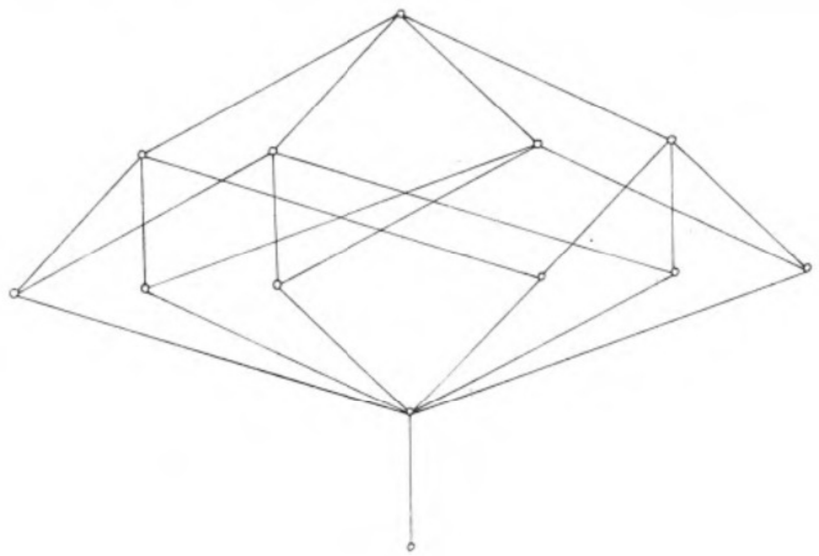
$$(N) \qquad a \wedge b \prec a \Rightarrow b \prec a \vee b$$

and the dual neighborhood condition

$$(N^*) \qquad b \prec a \vee b \Rightarrow a \wedge b \prec a.$$

It is also well-known that (N) and (N^*) together do not imply modularity. The neighborhood condition (N) implies the covering condition
(C) b arbitrary, p atom and $b \wedge p = 0 \Rightarrow b \prec b \vee p$ and the dual neighborhood condition (N^*) implies the dual covering condition
(C^*) b arbitrary, m dual atom and $b \vee m = 1 \Rightarrow b \wedge m \prec b$.
It is easy to see that even for finite lattices the conditions (C) and (C^*) together do not imply (N) or (N^*).

From now on we consider algebraic lattices satisfying the covering condition (C) and the dual neighborhood condition (N^*). Our aim is to characterize in these lattices the property of being atomistic by means of neat elements. Before doing this in the next section we give an example of a finite lattice which shows that (C) and (N^*) together imply neither modularity nor (N):



3. A characterization of neatness in algebraic lattices with (C) and (N^*)

In this section we prove the following

Theorem. *Let L be an algebraic lattice with* (C) *and* (N^*). *Then the following conditions are equivalent:*

(i) *L is atomistic*
(ii) *Every element of L is pure in L*
(iii) *Every element of L is neat in L.*

Proof. (i)$\Rightarrow$(ii): An atomistic lattice with (C) is a so-called AC-lattice. An AC-lattice with (N^*) is also called finite-modular (s. [9, Section 9]). An algebraic finite-modular AC-lattice is modular by [9, Theorem 14.1, p. 58]. By [6, Satz 2] in a modular algebraic lattice L the property of being atomistic is equivalent to the property that every element of L is pure in L.

(ii)$\Rightarrow$(iii): This implication clearly holds true since a pure element of an algebraic lattice is a fortiori a neat element.

(iii)$\Rightarrow$(i): We show first that L is atomic, i.e. that every interval $[0, b]$ ($b\in L$; $b\neq 0$) contains an atom. Since L is algebraic it follows by [1, Theorem 2.2, p. 14] that L is weakly atomic, that is, there exist in particular elements $u, v\in L$ such that

(1) $$0 \leqq u \prec v \leqq b.$$

By assumption the element u is neat in L. This means that there exists an element $p\in L$ such that

(2) $$p\vee u = v$$

and

(3) $$p\wedge u = 0.$$

Now (1) and (2) yield

$$u\prec v = u\vee p.$$

From this it follows by (3) and (N^*) that

$$0 = u\wedge p\prec p.$$

Hence p is an atom of L. Since $p\leqq b$ it follows that L is atomic.

Denote now by p_α ($\alpha\in A$) the family of all atoms $\leqq b$ (since L is atomic, this family is nonempty). Suppose that

(4) $$\vee(p_\alpha;\ \alpha\in A) < b.$$

By weak atomicity of L there exist elements $x, y\in L$ such that

$$\vee(p_\alpha\colon \alpha\in A) \leqq x\prec y \leqq b.$$

Since x is neat by assumption, we get the existence of an element $q\in L$ such that

$$x\prec y = x\vee q \quad \text{and} \quad x\wedge q = 0.$$

Again, q is an atom by (N^*) and $q\leqq b$. But q cannot occur among the $(p_\alpha\colon \alpha\in A)$

since $x \wedge p_\alpha = p_\alpha$ for all $\alpha \in A$. Thus our assumption (4) was false, i.e. every element ($\neq 0$) of L is the join of the atoms contained in it.

Corollary. *Let L be an algebraic lattice with* (C) *and* (N^*). *If one of the conditions* (i)—(iii) *of the preceding theorem is satisfied, then L is modular.*

PROOF. The assertion follows from the proof of the preceding theorem.

4. Concluding remarks

In [8] algebraic lattices with (C) and (C^*) have been considered. It was proved in [8] that in an algebraic lattice L with (C) and (C^*) the property of being atomistic implies that every element of L is pure in L. In connection with this A. Kertész raised the following question which can be found in [8]: Let L be an algebraic lattice with (C) and (C^*) and suppose that every element of L is pure in L. Is L then atomistic?

If L is of finite length, this question has an affirmative answer (s. [7]). By the theorem proved in this note the above mentioned question is answered affirmatively provided that the dual covering condition (C^*) is replaced by the stronger dual neighborhood condition (N^*). The original question remains open.

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