A generalized model of the Latin square design I.

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1. Introduction

In his paper [2] BÉLA GYIRES proved the following criterium for the randomized block design ([2], p. 285, Theorem 2).

The expectations of the sample elements can be decomposed into the sum of two quantities corresponding to the block-effect and to the treatment-effect, respectively, if and only if the expectations of the random errors are zero.

The proof was given by using Theorem 1 of [2] (p. 278), which was first proved by Corollary I of [4] (p. 213) and in a later paper [3] it was shown with a different method ([5], p. 401, Satz 233).

On the basis of the results obtained Professor B. Gyires called my attention to the investigation of the corresponding problem in the case of the Latin square design. Thus I should like to express my best thanks to him.

In this paper we will use the following notations: ξ, $\varepsilon(\xi_{ijh}, \varepsilon_{ijh})$ random variables; $\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_h, \boldsymbol{\zeta}, \ldots$ matrix-valued random variables, that is square matrices of order m consisting of random variables;

$\mathbf{I}, \mathbf{P}, \mathbf{\Gamma}, \mathbf{A}, \mathbf{C}, \mathbf{D}, \ldots$ square matrices of order m;
$\mathbf{O}, \mathbf{E}, \mathbf{\Omega}$ zero-, identity and primitive cyclic matrices (generally of order m);
$\mathbf{A}^*$ is the transpose of $\mathbf{A}$;
$\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$;

$M(\xi)$, $M(\boldsymbol{\xi})$ expectations of ξ and $\boldsymbol{\xi}$, respectively. $M(\boldsymbol{\xi})$ consists of the expectations of the elements of $\boldsymbol{\xi}$;

$\mathbf{A}=\begin{pmatrix} a_{11}, & \ldots, a_{1m} \\ \vdots & \\ a_{m1}, & \ldots, a_{mm} \end{pmatrix}$ is a matrix given by its elements;

$\mathbf{A}=\|a_{ij}\|_{m\times m}$ or $\mathbf{A}=\|a_{ij}\|_{i,j=\overline{1,m}}$ is a square matrix which is given by its general element;

$\mathbf{a}_0, \boldsymbol{\lambda}, \boldsymbol{\gamma}, \mathbf{0}, \ldots$ m-dimensional column-vectors ($\mathbf{0}$ is the zero vector);
$\mathbf{a}_0^*$ is an m-dimensional row-vector;
instead of $i=1, 2, \ldots, m$ we use the notation $i=\overline{1, m}$.

In the first part of our paper such a generalized model of the Latin square design is defined in which the generalizations of the theorems valid in the usual model can be proved. We constructed our model to examine the reversibility of the well-known theorem by which the expectations of the random errors are zero, if the expectations of the sample elements decompose into the sum of three quantities corresponding to the row-effect, the column-effect and the effect of treatment, respectively. The last problem will be investigated in the second part of our paper.

While in the case of the randomized blocks the proof of the above-mentioned criterium was equivalent to the determination of the general solution of the homogeneous matrix equation $\mathbf{AXB}^*=\mathbf{O}$ (see [2], p. 278, Theorem 1), in the case of Latin square design — as can be seen from the formulae (37), (38), (39) and (40) — the general solution of the homogeneous matrix equation

$$(\mathbf{E}-\mathbf{P})\mathbf{X}(\mathbf{E}-\mathbf{P})+\mathbf{PXP}-\frac{1}{m}\sum_{l=1}^{m}\mathbf{F}^l\mathbf{X}\mathbf{G}^l=\mathbf{O}$$

must be determined, where $\mathbf{F}$ and $\mathbf{G}$ are generally non-symmetric orthogonal square matrices for which $\sum_{l=1}^{m}\mathbf{F}^l=\sum_{l=1}^{m}\mathbf{G}^l=\mathbf{I}, \mathbf{F}^m=\mathbf{G}^m=\mathbf{E}$. (The definitions of $\mathbf{I}$ and $\mathbf{P}$ are given under (7).) But this complicated matrix equation can be obtained only for special — totally symmetric —, cyclic-, symmetric and for the Latin square which can be transformed into the symmetric standard form — Latin square designs. (Their definitions are given under (25)—(30) for the case of order 4.)

If we select a non-special Latin square then the term which corresponds to $\frac{1}{m}\sum_{l=1}^{m}\mathbf{F}^l\mathbf{X}\mathbf{G}^l$ in the homogeneous matrix equation will be very complicated.

In the second paragraph we give a short summary of the knowledge most important for us about the Latin square design. In paragraph 3 we shall introduce a natural generalization of the usual model of Latin square design with the formula (12) or (41). If $m=1$ then (1) follows from (41) and the corresponding expression for $M(\xi_{ijh})$ from (12). In paragraph 3 the decomposition of $\boldsymbol{\eta}_h$ ($\boldsymbol{\eta}_h$ is defined by (10)), using the matrix ξ and certain orthogonal matrices, is also important for special Latin square designs (see the formulae (31)—(34)), since the proofs of the theorems 2 and 5 are based upon these decompositions.

The theorems 1, 1′ and 2 are generalizations of the theorems valid in the usual model. Theorems 3, 4 and 5 (criteria) are in connection with the testing of statistical hypotheses (see Remark 6.). While these theorems hold for arbitrary Latin square designs, the theorems 2 and 5 are true only for the special Latin squares. Theorems 2 and 5 really consist of three theorems. The theorems 2, 3, 4 and 5 are proved by means of linear algebra using the properties of the cyclic, orthogonal and $\boldsymbol{\Gamma}$ matrices (the definition of $\boldsymbol{\Gamma}$ can be found under (12)), where $\boldsymbol{\Gamma}$ is determined unambigously by the given Latin square and applying the following theorem which is valid for a stochastic matrix. A matrix with non-negative elements is a stochastic one if and only if 1 is one of its eigenvalues and all components of the right eigenvector belonging to the eigenvalue 1 are equal to 1. From the theorems formulated and proved in Paragraph 3 it can be seen that our generalized assumption — decomposition (12) — which corresponds to the initial condition of the usual Latin square arrangement, is one of the fundamental requirements for each theorem.

The usual restrictions $\sum\lambda_i=\sum\nu_j=\sum\gamma_h=0$ were not used in the proofs of theorems 2, 3, 4 and 5. (The meaning of the notations λ_i, ν_j and γ_h can be found at (1).)

2. The Latin square method (design)

With the Latin square design it is assumed on the one hand that three factors — the row-effect, the column-effect and the effect of treatment — play a role in the formation of the random variable and on the other it is supposed that each factor occupies the level of the same number (let the number of levels be m) and there are no interactions between the three factors. Therefore, any level-triplet (i, j, h) where i is one of the levels of the row-effect, j is one of the levels of the column-effect, but h is one of the levels of the treatment-effect corresponding to the i-th row-effect and the j-th column-effect. (i, j, h) is frequently called a cell and satisfies certain typical conditions ([7], p. 229).

With a Latin square arrangement of level m, where the number of cells is m^2, let ξ_{ijh} be the result of the experiment which belongs to the level-triplet (i, j, h). From the foregoing it is clear that

$$\xi_{ijh} = \mu + \lambda_i + \nu_j + \gamma_h + \varepsilon_{ijh}, \tag{1}$$

where μ is a constant, the quantities λ_i, ν_j and γ_h correspond to the i-th row-effect, to the j-th column-effect and to the h-th treatment-effect, respectively. The random variables ε_{ijh} $(1 \leqq i, j, h = h(i, j) \leqq m)$ are assumed to be independent, normally distributed with parameters 0 and σ^2.

We assume that there exist expectations for the random variables ξ_{ijh} $(1 \leqq i, j, h(i, j) \leqq m)$. Let the total mean of the random variables ξ_{ijh} $(i = \overline{1, m};\ j = \overline{1, m})$ be

$$\bar{\bar{\xi}} = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \xi_{ijh}. \tag{2}$$

For the means of the i-th row, the j-th column and the h-th treatments let us introduce the following respective notations:

$$\bar{\bar{\xi}}_{i..} = \frac{1}{m} \sum_{j=1}^{m} \xi_{ijh}, \tag{3}$$

$$\bar{\bar{\xi}}_{.j.} = \frac{1}{m} \sum_{i=1}^{m} \xi_{ijh}, \tag{4}$$

$$\bar{\bar{\xi}}_{..h} = \frac{1}{m} \sum_{\substack{(i,j) \\ h = h(i,j)}} \xi_{ijh}. \tag{5}$$

On the right-side of (5) the sample elements must be summed for each pair (i, j) for which $h = h(i, j)$.

We call the differences $\bar{\bar{\xi}}_{i..} - \bar{\bar{\xi}}$ $(i = \overline{1, m})$,

$$\bar{\bar{\xi}}_{.j.} - \bar{\bar{\xi}} (j = \overline{1, m}) \quad \text{and} \quad \bar{\bar{\xi}}_{..h} - \bar{\bar{\xi}} (h = \overline{1, m})$$

discrepancies between rows, discrepancies between columns and discrepancies between treatments, respectively. The quantity

$$\xi_{ijh} - \bar{\bar{\xi}}_{i..} - \bar{\bar{\xi}}_{.j.} - \bar{\bar{\xi}}_{..h} + 2\bar{\bar{\xi}}$$

is the random error.

It is easy to prove the equality

$$\xi_{ijh}-\bar{\xi}_{i..}-\bar{\xi}_{.j.}-\bar{\xi}_{..h}+2\bar{\xi} = \varepsilon_{ijh}-\bar{\varepsilon}_{i..}-\bar{\varepsilon}_{.j.}-\bar{\varepsilon}_{..h}+2\bar{\varepsilon}, \tag{6}$$

where the quantities on the right side can be determined from the random variables ε_{ijh} in the same way as the quantities on the left side from the random variables ξ_{ijh} ([7], pp. 229—231).

It can be seen from (6) that in a Latin square design, that is if ξ_{ijh} can be decomposed into the form (1),

$$M(\xi_{ijh}-\bar{\xi}_{i..}-\bar{\xi}_{.j.}-\bar{\xi}_{..h}+2\bar{\xi}) = 0,$$

namely the expectations of the random errors equal zero.

3. A generalized model of the Latin square design

Let $\boldsymbol{\xi}=\|\xi_{ijh}\|_{i,j=\overline{1,m}}$ be the square matrix consisting of the random variables ξ_{ijh} which are defined with (1) and have expectations. Let the matrices with the identical constant elements 1 and m^{-1} be

$$\mathbf{I} = \|1\|_{m\times m}, \quad \mathbf{P} = \|m^{-1}\|_{m\times m}. \tag{7}$$

(**P** is a stochastic matrix). Let us define the matrix-valued random variables

$$\boldsymbol{\eta}_1 = \mathbf{P}\boldsymbol{\xi}, \quad \boldsymbol{\eta}_2 = \boldsymbol{\xi}\mathbf{P} \quad \text{and} \quad \boldsymbol{\zeta} = \mathbf{P}\boldsymbol{\xi}\mathbf{P}. \tag{8}$$

These with their elements in the intersections of their i-th row and j-th column may be written in the form

$$\boldsymbol{\eta}_1 = \|\bar{\xi}_{.j.}\|, \quad \boldsymbol{\eta}_2 = \|\bar{\xi}_{i..}\| \quad \text{and} \quad \boldsymbol{\zeta} = \|\bar{\xi}\|. \tag{9}$$

The matrix $\boldsymbol{\eta}_h$ built up from the means of the treatments is given by the equality

$$\boldsymbol{\eta}_h = \|\bar{\xi}_{..h(i,j)}\|_{i,j=\overline{1,m}}. \tag{10}$$

Remark 1. $\boldsymbol{\eta}_h$ — corresponding to the randomly selected Latin square of level m — contains each of the m different means of the treatments in each row and in each column exactly once.

The matrices

$$\begin{aligned} \boldsymbol{\eta}_1-\boldsymbol{\zeta} &= \|\bar{\xi}_{.j.}-\bar{\xi}\|_{m\times m}, \\ \boldsymbol{\eta}_2-\boldsymbol{\zeta} &= \|\bar{\xi}_{i..}-\bar{\xi}\|_{m\times m}, \\ \boldsymbol{\eta}_h-\boldsymbol{\zeta} &= \|\bar{\xi}_{..h(i,j)}-\bar{\xi}\|_{m\times m} \end{aligned} \tag{11}$$

and

$$\boldsymbol{\xi}-\boldsymbol{\eta}_1-\boldsymbol{\eta}_2-\boldsymbol{\eta}_h+2\boldsymbol{\zeta} = \|\xi_{ijh}-\bar{\xi}_{i..}-\bar{\xi}_{.j.}-\bar{\xi}_{..h}+2\bar{\xi}\|_{m\times m}$$

are named — on the basis of their elements — the matrix of discrepancies between columns, the matrix of discrepancies between rows and the matrix of discrepancies between treatments respectively, whereas the last matrix is the random error matrix.

In consequence of (1)

$$M(\boldsymbol{\xi}) = \mu \mathbf{a}_0 \mathbf{a}_0^* + \boldsymbol{\lambda} \mathbf{a}_0^* + \mathbf{a}_0 \mathbf{v}^* + \boldsymbol{\Gamma}, \tag{12}$$

where $\mathbf{a}_0$ is a column-vector which consists only of elements 1, $\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}$ is the column-vector of the row-effects, $\mathbf{v}^* = (\nu_1, \nu_2, \ldots, \nu_m)$ is the row-vector of the column-effects, $\boldsymbol{\Gamma}$ is the matrix from the different treatment-effects.

For a stochastic matrix $\mathbf{P}$

$$\mathbf{P}\mathbf{a}_0 = 1 \cdot \mathbf{a}_0, \tag{13}$$

therefore, the vector $\mathbf{a}_0$ in (12) may be considered as a right eigenvector of $\mathbf{P}$ which belongs to its eigenvalue 1 ([1], II. 73).

Remark 2. From (12)

$$M(\xi_{ijh}) = \mu + \lambda_i + \nu_j + \gamma_h$$

in the case of $m=1$ for each element of the matrix $\boldsymbol{\xi}$, which is generally assumed in the Latin square method. Therefore, the model which is given by the decomposition (12) may be regarded as a generalization of the model defined by (1).

Now we want to express the random error matrix occurring in (11) by the help of $\mathbf{P}$, $\boldsymbol{\xi}$ and certain orthogonal matrices in the case when the standard form of the randomly selected Latin square is symmetric, that is if the Latin square is cyclic, symmetric and total symmetric. We shall discuss the cases of Latin squares of levels (orders) 2, 3 and 4 in detail.

Comparing the form (11) of the random error matrix with (8) one can see that only $\boldsymbol{\eta}_h$ must be given by $\boldsymbol{\xi}$ and certain orthogonal matrices to get the above —mentioned form of the random error matrix.

Let $m=2$. Let us suppose that the treatments h_1, h_2 are applied according to the Latin squares of order 2

$$\begin{matrix} h_1 & h_2 \\ h_2 & h_1 \end{matrix} \quad \text{and} \quad \begin{matrix} h_2 & h_1 \\ h_1 & h_2. \end{matrix}$$

Let us give these Latin squares in the form

$$\begin{matrix} i_1 & i_2 \\ i_2 & i_1, \end{matrix} \tag{14}$$

where (i_1, i_2) represents a permutation of the indices of the treatments h_1, h_2. (14) is symmetric and cyclic (self-conjugate). In this case we shall prove that

$$\boldsymbol{\eta}_h = \frac{1}{2}(\boldsymbol{\xi} + \boldsymbol{\Omega}\boldsymbol{\xi}\boldsymbol{\Omega}), \tag{15}$$

where $\boldsymbol{\Omega} = \begin{pmatrix} 0, 1 \\ 1, 0 \end{pmatrix}$, $\boldsymbol{\Omega}^2 = \mathbf{E}$ ($\mathbf{E}$ is the identity matrix and $\boldsymbol{\Omega}$ is the primitive cyclical matrix).

PROOF OF *(15)*. According to the definition of ξ

$$\xi=\begin{pmatrix}\xi_{11i_1}, & \xi_{12i_2}\\ \xi_{21i_2}, & \xi_{22i_1}\end{pmatrix}. \tag{16}$$

On the basis of (10)

$$\eta_h=\begin{pmatrix}\bar{\xi}_{..i_1}, & \bar{\xi}_{..i_2}\\ \bar{\xi}_{..i_2}, & \bar{\xi}_{..i_1}\end{pmatrix}. \tag{17}$$

From this

$$\eta_h=\frac{1}{2}\begin{pmatrix}11i_1+22i_1, & 12i_2+21i_2\\ 21i_2+12i_2, & 22i_1+11i_2\end{pmatrix},$$

if we write in (17) the means of the treatments according to (5) and take into consideration only the indices of the random variables. Consequently $2\eta_h$ is the sum of matrices

$$\begin{pmatrix}11i_1, & 12i_2\\ 21i_2, & 22i_1\end{pmatrix} \tag{18}$$

and

$$\begin{pmatrix}22i_1, & 21i_2\\ 12i_2, & 11i_1\end{pmatrix}. \tag{19}$$

But (18) is identical with (16) and if we multiply (19) by $\Omega=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ from left and right, then we also get (16), that is

$$\Omega\begin{pmatrix}22i_1, & 21i_2\\ 12i_2, & 11i_1\end{pmatrix}\Omega=\xi.$$

Hence — since Ω is an orthogonal matrix ($\Omega^{-1}=\Omega^*=\Omega$) —

$$\begin{pmatrix}22i_1, & 21i_2\\ 12i_2, & 11i_1\end{pmatrix}=\Omega\xi\Omega.$$

Finally (17) can be written in the form (15), indeed.

Let now $m=3$. Then, as is well-known, there exists a standard Latin square which can be given in the form

$$\begin{matrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{matrix} \tag{20}$$

with the help of our notation introduced. Such a symmetric Latin square is named *self-conjugate*. From (20) we can get all Latin squares of order 3 by the permutation of the rows and columns. But these can be divided into two groups.

1. If the Latin square is symmetric, that is of form

$$\begin{matrix}i_1 & i_2 & i_3\\ i_2 & i_3 & i_1\\ i_3 & i_1 & i_2\end{matrix} \tag{21}$$

then — with the method used in the case of $m=2$ — the formula

$$\eta_h = \frac{1}{3}\sum_{l=1}^{3} \Omega^l \xi \Omega^l \tag{22}$$

can be obtained, where Ω is the primitive cyclical matrix of order 3, for which $\Omega^3 = \mathbf{E}$ ($\mathbf{E}$ is the identity matrix of order 3), $\Omega^2 = \Omega^*$ and $\Omega^* = \Omega^{-1}$. Finally $\Omega^2 = \Omega^{-1}$.

2. The Latin square of level 3 can still be cyclic:

$$\begin{matrix} i_1 & i_2 & i_3 \\ i_3 & i_1 & i_2 \\ i_2 & i_3 & i_1 \end{matrix} \tag{23}$$

This can be derived from (21) by the permutation of the second and third rows. Then

$$\eta_h = \frac{1}{3}\sum_{l=1}^{3} \Omega^l \xi (\Omega^l)^*. \tag{24}$$

Let us examine the case $m=4$. Then there are 4 standard Latin squares, which can be divided into two transformation sets ([6], pp. 108—109). The corresponding decompositions of the matrices η_h on the basis of their standard forms

$$\begin{matrix} i_1 & i_2 & i_3 & i_4 \\ i_2 & i_1 & i_4 & i_3 \\ i_3 & i_4 & i_2 & i_1 \\ i_4 & i_3 & i_1 & i_2 \end{matrix} \quad \text{(symmetric case);} \tag{25}$$

$$\begin{matrix} i_1 & i_2 & i_3 & i_4 \\ i_2 & i_3 & i_4 & i_1 \\ i_3 & i_4 & i_1 & i_2 \\ i_2 & i_1 & i_2 & i_3 \end{matrix} \quad \text{(totally symmetric case);} \tag{26}$$

$$\begin{matrix} i_1 & i_2 & i_3 & i_4 \\ i_4 & i_1 & i_2 & i_3 \\ i_3 & i_4 & i_1 & i_2 \\ i_2 & i_3 & i_4 & i_1 \end{matrix} \quad \text{(cyclic case);} \tag{27}$$

$$\begin{matrix} i_1 & i_2 & i_3 & i_4 \\ i_2 & i_4 & i_1 & i_3 \\ i_3 & i_1 & i_4 & i_2 \\ i_4 & i_3 & i_2 & i_1 \end{matrix} \quad \text{(symmetrical Latin square);} \tag{28}$$

$$\begin{matrix} i_1 & i_2 & i_3 & i_4 \\ i_2 & i_1 & i_4 & i_3 \\ i_3 & i_4 & i_1 & i_2 \\ i_4 & i_3 & i_2 & i_1 \end{matrix} \quad \text{(doubly symmetric case)} \tag{29}$$

are as follows:

$$(25') \qquad \boldsymbol{\eta}_h = \frac{1}{4}\sum_{l=1}^{4} \mathbf{C}^l \xi \mathbf{C}^l, \quad \text{where} \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$\mathbf{C}^4 = \mathbf{E}$; $\mathbf{E}$ is the identity matrix of order 4;

$$(26') \qquad \boldsymbol{\eta}_h = \frac{1}{4}\sum_{l=1}^{4} \boldsymbol{\Omega}^l \xi \boldsymbol{\Omega}^l;$$

$$(27') \qquad \boldsymbol{\eta}_h = \frac{1}{4}\sum_{l=1}^{4} \boldsymbol{\Omega}^l \xi (\boldsymbol{\Omega}^l)^*,$$

$\boldsymbol{\Omega}$ is the primitive cyclic matrix of order 4;

$$(28') \qquad \boldsymbol{\eta}_h = \frac{1}{4}\sum_{l=1}^{4} \mathbf{B}^l \xi \mathbf{B}^l,$$

and for the orthogonal matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{B}^4 = \mathbf{E}$ is valid;

$$(29') \qquad \begin{aligned} \boldsymbol{\eta}_h = \frac{1}{4}\Bigg(\xi + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xi \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \\ + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xi \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \xi \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Bigg), \end{aligned}$$

and the constant matrices in (29′) are orthogonal and involutory.

The general Latin square

$$(30) \qquad \begin{matrix} i_1 & i_4 & i_3 & i_2 \\ i_4 & i_2 & i_1 & i_3 \\ i_2 & i_3 & i_4 & i_1 \\ i_3 & i_1 & i_2 & i_4 \end{matrix}$$

can be transformed into the standard form (25). Then

$$(30') \qquad \boldsymbol{\eta}_h = \frac{1}{4}\sum_{l=1}^{4} \boldsymbol{\Omega}^l \xi \mathbf{B}^l$$

where $\mathbf{B}$ is identical to the matrix in (28′).

In the case of $m=5$ the number of the standard squares is 56 ([6], pp. 110—

111). In the first transformation set there are no self-conjugate (symmetric) standard Latin squares. The matrices $\boldsymbol{\eta}_h$ which are determined by such Latin squares cannot be written in the former way. In the case of the self-conjugate squares of the second transformation set the matrices $\boldsymbol{\eta}_h$ have also representations with $\boldsymbol{\xi}$ and orthogonal matrices.

Generally for a totally symmetric Latin square of order (level) m

$$\boldsymbol{\eta}_h = \frac{1}{m}\sum_{l=1}^{m} \boldsymbol{\Omega}^l \boldsymbol{\xi} \boldsymbol{\Omega}^l; \tag{31}$$

in the cyclic case

$$\boldsymbol{\eta}_h = \frac{1}{m}\sum_{l=1}^{m} \boldsymbol{\Omega}^l \boldsymbol{\xi} (\boldsymbol{\Omega}^l)^*, \tag{32}$$

$\boldsymbol{\Omega}$ is the primitive cyclic matrix of order m; for a symmetric (self-conjugate) Latin square

$$\boldsymbol{\eta}_h = \frac{1}{m}\sum_{l=1}^{m} \mathbf{A}^l \boldsymbol{\xi} \mathbf{A}^l, \tag{33}$$

where $\mathbf{A}$ is an orthogonal matrix of order m for which $\mathbf{A}^m=\mathbf{E}$, $\sum_{l=1}^{m} \mathbf{A}^l=\mathbf{I}$, $\mathbf{E}$ is the identity matrix of order m, $\mathbf{I}$ is the matrix of order m whose elements are all 1;

for the Latin square of order m which can be transformed into a symmetric standard from

$$\boldsymbol{\eta}_h = \frac{1}{m}\sum_{l=1}^{m} \mathbf{A}^l \boldsymbol{\xi} \mathbf{D}^l, \tag{34}$$

$\mathbf{A}$ and $\mathbf{D}$ are orthogonal matrices of order m for which $\mathbf{A}^m=\mathbf{D}^m=\mathbf{E}$, further $\sum_{l=1}^{m} \mathbf{A}^l=\sum_{l=1}^{m} \mathbf{D}^l=\mathbf{I}$.

According to (8) and the last formula of (11) the random error matrix can be given in the form

$$(\mathbf{E}-\mathbf{P})\boldsymbol{\xi}(\mathbf{E}-\mathbf{P})+\mathbf{P}\boldsymbol{\xi}\mathbf{P}-\boldsymbol{\eta}_h \tag{35}$$

where $\mathbf{P}$ is defined under (7).

So the matrix equation

$$M(\boldsymbol{\xi}-\boldsymbol{\eta}_1-\boldsymbol{\eta}_2-\boldsymbol{\eta}_h+2\boldsymbol{\zeta}) = \mathbf{O} \tag{36}$$

which is corresponding to the last formula in Paragraph 2 in our generalized model, using the formulae (31)—(34), can be written in the following forms if we introduce the notation $\hat{M}(\boldsymbol{\xi})=(\mathbf{E}-\mathbf{P})M(\boldsymbol{\xi})(\mathbf{E}-\mathbf{P})+\mathbf{P}M(\boldsymbol{\xi})\mathbf{P}$

$$\hat{M}(\boldsymbol{\xi})-\frac{1}{m}\sum_{l=1}^{m} \boldsymbol{\Omega}^l M(\boldsymbol{\xi})\boldsymbol{\Omega}^l = \mathbf{O} \tag{37}$$

(the Latin square is totally symmetric);

$$\hat{M}(\boldsymbol{\xi})-\frac{1}{m}\sum_{l=1}^{m} \boldsymbol{\Omega}^l M(\boldsymbol{\xi})(\boldsymbol{\Omega}^l)^* = \mathbf{O} \tag{38}$$

(for the cyclic Latin square);

$$\hat{M}(\xi)-\frac{1}{m}\sum_{l=1}^{m}\mathbf{A}^{l}M(\xi)\mathbf{A}^{l}=\mathbf{O} \tag{39}$$

(the Latin square is symmetric);

$$\hat{M}(\xi)-\frac{1}{m}\sum_{l=1}^{m}\mathbf{A}^{l}M(\xi)\mathbf{D}^{l}=\mathbf{O} \tag{40}$$

(if the Latin square can be transformed into the symmetric standard form).

Theorem 1. *If all the elements of the square matrix* $\xi=\|\xi_{ijh}\|_{i,j=\overline{1,m}}$ *are of the form* (1) *and the random variables* ε_{ijh} *have zero expectations, then the expectation of the random error matrix is the zero matrix.*

PROOF. The proof of the statement is trivial on the basis of (6) and the last formula of (11).

Remark 3. If (1) is valid for every element of ξ and $M(\varepsilon_{ijh})=0$ $(i,j=\overline{1,m})$, then

$$\xi=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\mathbf{\Gamma}+\boldsymbol{\varepsilon}, \tag{41}$$

where $\boldsymbol{\varepsilon}=\|\varepsilon_{ijh}\|_{i,j=\overline{1,m}}$ and $M(\boldsymbol{\varepsilon})=\mathbf{O}$. (The exact meaning of the quantities in (41) can be found after the formula (12).)

Conversely, the conditions of Theorem 1 follow from (41). Hence the assumptions of Theorem 1 are equivalent to (41). Therefore Theorem 1 can be formulated also in the following way.

Theorem 1'. *If for the matrix* ξ *the decomposition* (41) *is valid, then the expectation of the random error matrix is the zero matrix.*

According to the following theorem in our generalized model of the Latin square design — in the case of certain special Latin squares — the conditions of Theorem 1 can be replaced by (12).

Theorem 2. *If the Latin square is a total symmetric, cyclic or a symmetric one, moreover if* $M(\xi)$ *exists and*

$$M(\xi)=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\mathbf{\Gamma},$$

then

$$M(\xi-\boldsymbol{\eta}_1-\boldsymbol{\eta}_2-\boldsymbol{\eta}_h+2\zeta)=\mathbf{O}.$$

PROOF. I. For a total symmetric Latin square (36) can be rewritten in the form (37). We shall show that the matrices $\mu\mathbf{a}_0\mathbf{a}_0^*$, $\boldsymbol{\lambda}\mathbf{a}_0^*$, $\mathbf{a}_0\mathbf{v}^*$ and $\mathbf{\Gamma}$ — the members of the decomposition of $M(\xi)$ — satisfy (37).

The left side of (37) after the substitution of $\mu\mathbf{a}_0\mathbf{a}_0^*$ and considering (13), as well as the eigenvalue equations

$$(\mathbf{E}-\mathbf{P})\mathbf{a}_0=0\cdot\mathbf{a}_0, \tag{42}$$

$$\mathbf{\Omega}^l\mathbf{a}_0=1\cdot\mathbf{a}_0 \quad (l=\overline{1,m-1}), \tag{43}$$

$$\mathbf{a}_0^*\mathbf{\Omega}^l=\mathbf{a}_0^* \quad (l=\overline{1,m-1}) \tag{44}$$

can be brought in the form

$$\mu \mathbf{a}_0 \mathbf{a}_0^* - \frac{\mu}{m} (\mathbf{a}_0 \mathbf{a}_0^* + \sum_{l=1}^{m-1} \mathbf{a}_0 \mathbf{a}_0^*),$$

and this expression evidently equals the zero matrix. Consequently $\mu \mathbf{a}_0 \mathbf{a}_0^*$ is a solution of (37).

$\boldsymbol{\lambda} \mathbf{a}_0^*$ also satisfies (37). By substitution of $\boldsymbol{\lambda} \mathbf{a}_0^*$ on the left side of (37), and taking into account (13), (42) and (44) we get the expression

$$P \boldsymbol{\lambda} \mathbf{a}_0^* - \frac{1}{m} (\boldsymbol{\lambda} \mathbf{a}_0^* + \sum_{l=1}^{m-1} \boldsymbol{\Omega}^l \boldsymbol{\lambda} \mathbf{a}_0^*),$$

and in a simpler form

$$[\mathbf{P} - \frac{1}{m} (\mathbf{E} + \sum_{l=1}^{m-1} \boldsymbol{\Omega}^l)] \boldsymbol{\lambda} \mathbf{a}_0^*,$$

which is equal to the zero matrix, since

$$\frac{1}{m} (\mathbf{E} + \sum_{l=1}^{m-1} \boldsymbol{\Omega}^l) = \mathbf{P}.$$

$\mathbf{a}_0 \mathbf{v}^*$ is also a solution of (37). Substituting $\mathbf{a}_0 \mathbf{v}^*$ for $M(\xi)$ on the left side of (37) and using the formulae (13), (42) and (43) we obtain

$$\mathbf{a}_0 \mathbf{v}^* [\mathbf{P} - \frac{1}{m} (\mathbf{E} + \sum_{l=1}^{m-1} \boldsymbol{\Omega}^l)],$$

which also equals the zero matrix, as the term in square brackets equals the zero matrix.

We shall only prove, that the total symmetric matrix

$$\boldsymbol{\Gamma} = \begin{pmatrix} \gamma_{i_1}, & \gamma_{i_2}, & \dots, & \gamma_{i_m} \\ \gamma_{i_2}, & \gamma_{i_3}, & \dots, & \gamma_{i_1} \\ \vdots \\ \gamma_{i_m}, & \gamma_{i_1}, & \dots, & \gamma_{i_{m-1}} \end{pmatrix}$$

is a solution of (37). Substituting it in the left side of (37) and using the easily provable relations

(45) $$\boldsymbol{\Gamma} \mathbf{P} = \mathbf{P} \boldsymbol{\Gamma};$$

(46) $$\mathbf{P}^2 = \mathbf{P};$$

47) $$\boldsymbol{\Gamma} \boldsymbol{\Omega}^l = (\boldsymbol{\Omega}^l)^* \boldsymbol{\Gamma} \quad (l = \overline{1, m-1});$$

(48) $$\boldsymbol{\Omega}^l (\boldsymbol{\Omega}^l)^* = \mathbf{E} \quad (l = 1, m-1)$$

(the matrices $\boldsymbol{\Omega}^l$ are orthogonal) it can be seen that the left side of (37) is equal to the zero matrix.

In this way Theorem 2 is proved for the total symmetric case.

II. Proof for the cyclic Latin square. We must show, that the matrices $\mu \mathbf{a}_0 \mathbf{a}_0^*$, $\boldsymbol{\lambda} \mathbf{a}_0^*$ and $\mathbf{a}_0 \mathbf{v}^*$ are solutions for the form (38) of the matrix equation (36).

In the proofs we shall utilize the fact that the multiplication of cyclic matrices is a commutative operation. Since matrices $\mu \mathbf{a}_0 \mathbf{a}_0^*$ and $\mathbf{\Gamma}$ are cyclic, therefore, it is enough to take into consideration only one of the two cases. Substituting $\mathbf{\Gamma}$ in the left side of (38) and considering the commutativity, for the left side we get the expression

$$(\mathbf{E}-\mathbf{P})^2\mathbf{\Gamma}+\mathbf{P}^2\mathbf{\Gamma}-\frac{1}{m}\sum_{l=1}^{m}\mathbf{\Omega}^l(\mathbf{\Omega}^l)^*\mathbf{\Gamma}.$$

Because of (46) and (48) the left side of (38) is

$$(\mathbf{E}-2\mathbf{P}+\mathbf{P}+\mathbf{P}-\mathbf{E})\mathbf{\Gamma},$$

which evidently equals the zero matrix. Hence $\mathbf{\Gamma}$ is a solution of (38).

By substitution of $\boldsymbol{\lambda}\mathbf{a}_0^*$ and taking into account (13), (42) and (43) the left side of (38) is

$$(\mathbf{P}-\frac{1}{m}\sum_{l=1}^{m}\mathbf{\Omega}^l)\boldsymbol{\lambda}\mathbf{a}_0^*$$

which also equals the zero matrix. Therefore $\boldsymbol{\lambda}\mathbf{a}_0^*$ is also a solution of (38).

Similarly, it can be seen that $\mathbf{a}_0\mathbf{v}^*$ is a solution of the matrix equation (38).

Remark 4. The formulae corresponding to (45) and (47) are also valid in the case of a cyclic Latin square. Let the $\mathbf{\Gamma}$ corresponding to a cyclic Latin square be denoted by $\mathbf{\Gamma}_c$. For $\mathbf{\Gamma}_c$

(49) $$\mathbf{\Gamma}_c\mathbf{P} = \mathbf{P}\mathbf{\Gamma}_c,$$

(50) $$\mathbf{\Gamma}_c\mathbf{\Omega}^l = \mathbf{\Omega}^l\mathbf{\Gamma}_c \quad (l=\overline{1, m-1}).$$

Let the matrix $\mathbf{\Gamma}$ belonging to a total symmetric Latin square be $\mathbf{\Gamma}_{ts}$. Since

(51) $$\mathbf{\Gamma}_{ts} = \mathbf{C}\mathbf{\Gamma}_c, \quad \text{where} \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \dots 0 & 0 \\ 0 & 0 & 0 \dots 0 & 1 \\ 0 & 0 & 0 \dots 1 & 0 \\ \vdots & & & \\ 0 & 1 & 0 \dots 0 & 0 \end{pmatrix}_{m\times m},$$

(52) $$\mathbf{PC} = \mathbf{CP}$$

and

(53) $$\mathbf{\Omega}^l\mathbf{C} = \mathbf{C}(\mathbf{\Omega}^l)^* \quad (l=\overline{1, m-1}),$$

hence we can get (45) and (47) also for $\mathbf{\Gamma}_{ts}$ if we multiply (49) and (50) by $\mathbf{C}$ from the left and consider the relations (51), (52) and (53).

Remark 5. (45) is true for a matrix $\mathbf{\Gamma}$ which belongs to an arbitrary Latin square. (47) is still certainly valid for a symmetric Latin square, as the orthogonal matrices of order m $\mathbf{A}^l$ $(l=\overline{1, m-1})$ occurring in (33) are permutation matrices and for their multiplications by symmetric matrices

(54) $$\mathbf{\Gamma}_s\mathbf{A}^l = (\mathbf{A}^l)^*\mathbf{\Gamma}_s \quad (l=\overline{1, m-1})$$

($\mathbf{\Gamma}_s$ belongs to a symmetric Latin square).

III. Proof for symmetric Latin square. In this case (36) can be rewritten in the form (39). We shall prove that $\mu \mathbf{a}_0 \mathbf{a}_0^*$, $\boldsymbol{\lambda} \mathbf{a}_0^*$, $\mathbf{a}_0 \mathbf{v}^*$ and $\boldsymbol{\Gamma}_s$ are solutions of (39). Since $\mu \mathbf{a}_0 \mathbf{a}_0^*$ and also $\boldsymbol{\Gamma}_s$ are symmetric matrices, therefore it is sufficient for example to prove that $\boldsymbol{\Gamma}_s$ satisfies (39). Substituting $\boldsymbol{\Gamma}_s$ for $M(\boldsymbol{\xi})$ on the left side of (39) and using (45) on the basis of Remark 5 and taking into account (54), the left side of (39) will be

$$(\mathbf{E}-\mathbf{P})^2\boldsymbol{\Gamma}_s+\mathbf{P}\boldsymbol{\Gamma}_s-\frac{1}{m}\sum_{l=1}^{m}\mathbf{A}^l\boldsymbol{\Gamma}_s\mathbf{A}^l,$$

or rather $\boldsymbol{\Gamma}_s-\frac{1}{m}\sum_{l=1}^{m}\mathbf{A}^l(\mathbf{A}^l)^*\boldsymbol{\Gamma}_s$ and this is evidently equal to the zero matrix.

We show that also $\boldsymbol{\lambda}\mathbf{a}_0^*$ is a solution for (39). Substituting $\boldsymbol{\lambda}\mathbf{a}_0^*$ for $M(\boldsymbol{\xi})$ on the left side of (39) and taking into consideration (13), (42) and

$$(\mathbf{A}^l)^*\mathbf{a}_0=1\cdot\mathbf{a}_0 \quad (l=\overline{1, m-1}) \tag{55}$$

for the left side of (39)

$$\mathbf{P}\boldsymbol{\lambda}\mathbf{a}_0^*-\frac{1}{m}\sum_{l=1}^{m}\mathbf{A}^l\boldsymbol{\lambda}\mathbf{a}_0^*$$

can be got. In a simpler form this is

$$\left(\mathbf{P}-\frac{1}{m}\mathbf{I}\right)\boldsymbol{\lambda}\mathbf{a}_0^*,$$

which equals the zero matrix.

It is also calculable in a similar way that $\mathbf{a}_0\mathbf{v}^*$ satisfies (39). Thus Theorem 2 completely is proved.

In our generalized model the following criterion is true.

Theorem 3. *If the expectation of the square matrix* $\boldsymbol{\xi}$ *exists and* $M(\boldsymbol{\xi})=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\boldsymbol{\Gamma}$, *moreover* $\mathbf{a}_0$ *is the right-eigenvector of the square matrix* $\mathbf{P}$, *which belongs to the eigenvalue* 1, *so* $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=\mathbf{O}$ ($\boldsymbol{\eta}_2-\boldsymbol{\zeta}$ *is defined by* (11)) *if and only if* $\boldsymbol{\lambda}=c\mathbf{a}_0$, *where c is a constant and* $\boldsymbol{\lambda}$ *is the column-vector of the row-effects.*

Proof *1.* First we prove that from $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=\mathbf{O}$ — on the assumptions of the theorem — follows $\boldsymbol{\lambda}=c\mathbf{a}_0$.

Because of (8) $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=(\mathbf{E}-\mathbf{P})M(\boldsymbol{\xi})\mathbf{P}$. From the conditions of our theorem (see (12) and (13)) and (42) the form

$$(\mathbf{E}-\mathbf{P})\boldsymbol{\lambda}\mathbf{a}_0^*+(\mathbf{E}-\mathbf{P})\boldsymbol{\Gamma}\mathbf{P}=\mathbf{O} \tag{56}$$

of $(\mathbf{E}-\mathbf{P})M(\boldsymbol{\xi})\mathbf{P}=\mathbf{O}$ can be got. In consequence of (46), Remark 5 and $\mathbf{P}^2=\mathbf{P}$ from (56)

$$(\mathbf{E}-\mathbf{P})\boldsymbol{\lambda}\mathbf{a}_0^*=\mathbf{O}. \tag{57}$$

Since $\mathbf{a}_0$ is a nonzero vector, therefore (57) is true then and only then if

$$(\mathbf{E}-\mathbf{P})\boldsymbol{\lambda}=\mathbf{0}$$

($\mathbf{0}$ is the m-dimensional zero column vector). Thus $\mathbf{P}\boldsymbol{\lambda}=\boldsymbol{\lambda}$. We see from this, in comparison with (13), that $\boldsymbol{\lambda}$ would be another (different from $\mathbf{a}_0$) right eigenvector of the stochastic matrix $\mathbf{P}$, which also belongs to the eigenvalue 1. But it is well-known that $\mathbf{P}$ has 1 as a simple eigenvalue belonging to the right eigenvector $\mathbf{a}_0$. Hence $\mathbf{P}\boldsymbol{\lambda}=\boldsymbol{\lambda}$ can be valid only then, if $\boldsymbol{\lambda}=c\mathbf{a}_0$, where c is a numerical parameter.

2. From $\boldsymbol{\lambda}=c\mathbf{a}_0$ also follows $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=\mathbf{O}$. Because of $\boldsymbol{\lambda}=c\mathbf{a}_0$

$$M(\boldsymbol{\xi}) = (\mu+c)\,\mathbf{a}_0\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\boldsymbol{\Gamma}$$

is realized.

Substituting this decomposition into $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=(\mathbf{E}-\mathbf{P})M(\boldsymbol{\xi})\mathbf{P}$ and taking into account (42) one can obtain that $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=(\mathbf{E}-\mathbf{P})\boldsymbol{\Gamma}\mathbf{P}$. In consequence of (45) and Remark 5 $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=[(\mathbf{E}-\mathbf{P})\mathbf{P}]\boldsymbol{\Gamma}$. Since $\mathbf{P}=\mathbf{P}^2$, therefore $M(\boldsymbol{\eta}_2-\boldsymbol{\zeta})=\mathbf{O}$.

The following theorem can be proved similarly to Theorem 3.

Theorem 4. *Let $\boldsymbol{\xi}$ be a square matrix of order m having the expectation of the form $M(\boldsymbol{\xi})=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*$ and let $\mathbf{P}\mathbf{a}_0=\mathbf{a}_0$. Then $M(\boldsymbol{\eta}_1-\boldsymbol{\zeta})=\mathbf{O}$* if and only if *$\mathbf{v}=d\mathbf{a}_0$, where d is a numerical parameter, $\mathbf{v}$ is the column-vector of the column-effects and $\boldsymbol{\eta}_1-\boldsymbol{\zeta}$ is the matrix of discrepancies between columns.*

The following theorem is true only for the Latin squares of special form.

Theorem 5. *If the Latin square is a total symmetric, cyclic or a symmetric one, moreover if the expectation $M(\boldsymbol{\xi})$ of the matrix $\boldsymbol{\xi}$ exists and $M(\boldsymbol{\xi})=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\boldsymbol{\Gamma}$ and $\mathbf{P}\mathbf{a}_0=\mathbf{a}_0$, then the equation $M(\boldsymbol{\eta}_h-\boldsymbol{\zeta})=\mathbf{O}$ holds if and only if*

$$\boldsymbol{\Gamma} = \bar{\gamma}\mathbf{a}_0\mathbf{a}_0^*,$$

where $\boldsymbol{\eta}_h-\boldsymbol{\zeta}$ is the matrix defined by (11),

$$\bar{\gamma} = \frac{1}{m}\sum_{i=1}^{m}\gamma_i$$

(the i-th treatment-effect is denoted by γ_i).

Proof. I. The total symmetric case.

A) The condition is necessary.

Let us substitute into $M(\boldsymbol{\eta}_h-\boldsymbol{\zeta})=\mathbf{O}$ the matrices $\boldsymbol{\eta}_h$ in the form (31) and $\boldsymbol{\zeta}$ on the basis of (8). Then

$$\frac{1}{m}\sum_{l=1}^{m}\boldsymbol{\Omega}^l M(\boldsymbol{\xi})\boldsymbol{\Omega}^l-\mathbf{P}M(\boldsymbol{\xi})\mathbf{P} = \mathbf{O}. \tag{58}$$

From (58) because of (12)

$$\mu\sum_{l=1}^{m}(\boldsymbol{\Omega}^l\mathbf{a}_0)(\mathbf{a}_0^*\boldsymbol{\Omega}^l)+\sum_{l=1}^{m}\boldsymbol{\Omega}^l\boldsymbol{\lambda}(\mathbf{a}_0^*\boldsymbol{\Omega}^l)+\sum_{l=1}^{m}(\boldsymbol{\Omega}^l\mathbf{a}_0)\mathbf{v}^*\boldsymbol{\Omega}^l+$$

$$+\sum_{l=1}^{m}\boldsymbol{\Omega}^l\boldsymbol{\Gamma}\boldsymbol{\Omega}^l-m[\mu\mathbf{P}\mathbf{a}_0(\mathbf{P}\mathbf{a}_0)^*+\mathbf{P}\boldsymbol{\lambda}(\mathbf{P}\mathbf{a}_0)^*+\mathbf{P}\mathbf{a}_0(\mathbf{P}\mathbf{v})^*+\mathbf{P}\boldsymbol{\Gamma}\mathbf{P}] = \mathbf{O}.$$

This equation can be rewritten into a simpler form using (13), (43), (44), (45), (46), (47) and (48):

$$\left(\sum_{l=1}^{m}\boldsymbol{\Omega}^l-m\mathbf{P}\right)\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*\left(\sum_{l=1}^{m}\boldsymbol{\Omega}^l-m\mathbf{P}\right)+(m\mathbf{E}-m\mathbf{P})\boldsymbol{\Gamma} = \mathbf{O}.$$

From this because of $\sum_{l=1}^{m} \mathbf{\Omega}^l = m\mathbf{P}$

$$m(\mathbf{E}-\mathbf{P})\mathbf{\Gamma} = \mathbf{O},$$

that is $\mathbf{\Gamma} = \mathbf{P\Gamma}$. Since $\mathbf{P} = \frac{1}{m}\mathbf{I}$, therefore, $\mathbf{\Gamma} = \frac{1}{m}\mathbf{I\Gamma}$. Thus — on the basis of the definition $\mathbf{\Gamma}$ and $\bar{\gamma} = \frac{1}{m}\sum_{i=1}^{m}\gamma_i$ —

$$\mathbf{\Gamma} = \bar{\gamma}\mathbf{I},$$

or

$$\mathbf{\Gamma} = \bar{\gamma}\mathbf{a}_0\mathbf{a}_0^*.$$

B) The condition is sufficient.

This can be ascertained so that we substitute $\mathbf{\Gamma} = \bar{\gamma}\mathbf{a}_0\mathbf{a}_0^*$ into (12). Then we can show that $M(\boldsymbol{\xi})$ — obtained in this way — satisfies the equation (58) using the equalities (13), (43) and (44).

II. The cyclic case.

1. Let us substitute into the $M(\boldsymbol{\eta}_h - \boldsymbol{\zeta}) = \mathbf{O}$ $\boldsymbol{\eta}_h$ from (32) and $\boldsymbol{\zeta}$ on the basis of (8) and take into account the decomposition (12). Thus $M(\boldsymbol{\eta}_h - \boldsymbol{\zeta}) = \mathbf{O}$ can be written in the form

$$(59) \qquad \mu\sum_{l=1}^{m}(\mathbf{\Omega}^l\mathbf{a}_0)(\mathbf{\Omega}^l\mathbf{a}_0)^* + \sum_{l=1}^{m}\mathbf{\Omega}^l\boldsymbol{\lambda}(\mathbf{\Omega}^l\mathbf{a}_0)^* +$$
$$+ \sum_{l=1}^{m}(\mathbf{\Omega}^l\mathbf{a}_0)\mathbf{v}^*(\mathbf{\Omega}^l)^* + \sum_{l=1}^{m}\mathbf{\Omega}^l\mathbf{\Gamma}(\mathbf{\Omega}^l)^* -$$
$$- m[\mu\mathbf{P}\mathbf{a}_0(\mathbf{P}\mathbf{a}_0)^* + \mathbf{P}\mathbf{a}_0(\mathbf{P}\mathbf{v})^* + \mathbf{P}\boldsymbol{\lambda}(\mathbf{P}\mathbf{a}_0)^* + \mathbf{P\Gamma P}] = \mathbf{O}.$$

(59) becomes simpler in consequence of (13), (43), (48), (49) and (50):

$$\left(\sum_{l=1}^{m}\mathbf{\Omega}^l - m\mathbf{P}\right)\boldsymbol{\lambda}\mathbf{a}_0^* + \mathbf{a}_0\mathbf{v}^*\left(\sum_{l=1}^{m}(\mathbf{\Omega}^l)^* - m\mathbf{P}\right) + m(\mathbf{E}-\mathbf{P})\mathbf{\Gamma} = \mathbf{O}.$$

Since the first and second expressions of the left side are zero matrices, the simplest form of (59) is

$$m(\mathbf{E}-\mathbf{P})\mathbf{\Gamma} = \mathbf{O}.$$

However, from this, as it was seen in the total symmetric case, follows the equality

$$\mathbf{\Gamma} = \bar{\gamma}\mathbf{a}_0\mathbf{a}_0^*,$$

that is the necessity of the condition for the cyclic Latin square.

2. Because of (12) and $\mathbf{\Gamma} = \bar{\gamma}\mathbf{a}_0\mathbf{a}_0^*$

$$M(\boldsymbol{\xi}) = (\mu + \bar{\gamma})\mathbf{a}_0\mathbf{a}_0^* + \boldsymbol{\lambda}\mathbf{a}_0^* + \mathbf{a}_0\mathbf{v}^*.$$

So the left side of (58) is

$$\mu\sum_{l=1}^{m}\mathbf{\Omega}^l\mathbf{a}_0(\mathbf{\Omega}^l\mathbf{a}_0)^* + \sum_{l=1}^{m}\mathbf{\Omega}^l\boldsymbol{\lambda}(\mathbf{\Omega}^l\mathbf{a}_0)^* + \sum_{l=1}^{m}\mathbf{\Omega}^l\mathbf{a}_0(\mathbf{\Omega}^l\mathbf{v})^* +$$
$$+\bar{\gamma}\sum_{l=1}^{m}\mathbf{\Omega}^l\mathbf{a}_0(\mathbf{\Omega}^l\mathbf{a}_0)^* - m[\mu\mathbf{P}\mathbf{a}_0(\mathbf{P}\mathbf{a}_0)^* + \mathbf{P}\mathbf{a}_0(\mathbf{P}\mathbf{v})^* + \mathbf{P}\boldsymbol{\lambda}(\mathbf{P}\mathbf{a}_0)^* + \bar{\gamma}\mathbf{P}\mathbf{a}_0(\mathbf{P}\mathbf{a}_0)^*].$$

This sum can be rewritten according to (13) and (43) in the form

$$\left(\sum_{l=1}^{m} \mathbf{\Omega}^l - m\mathbf{P}\right) \boldsymbol{\lambda}\mathbf{a}_0^* + \mathbf{a}_0 \mathbf{v}^* \left(\sum_{l=1}^{m} (\mathbf{\Omega}^l)^* - m\mathbf{P}\right)$$

and this expression eventually equals the zero matrix.

III. The case of the symmetric Latin square.

1. The expression of $\boldsymbol{\eta}_h$ with $\boldsymbol{\xi}$ is given by (33). In consequence of (33) and (8) $M(\boldsymbol{\eta}_h - \boldsymbol{\zeta}) = \mathbf{O}$ can be given in the form

$$\sum_{l=1}^{m} \mathbf{A}^l M(\boldsymbol{\xi}) \mathbf{A}^l - m\mathbf{P}M(\boldsymbol{\xi})\mathbf{P} = \mathbf{O}. \tag{60}$$

Substituting $M(\boldsymbol{\xi})$ into (60) on the basis of the representation (12) and using (13), (46), (54), (55) and the relations $\mathbf{A}^l(\mathbf{A}^l)^* = \mathbf{E}$ $(l = \overline{1, m})$

$$\mu \sum_{l=1}^{m} \mathbf{a}_0 \mathbf{a}_0^* + \sum_{l=1}^{m} \mathbf{A}^l \boldsymbol{\lambda} \mathbf{a}_0^* + \sum_{l=1}^{m} \mathbf{a}_0 \mathbf{v}^* \mathbf{A}^l + \sum_{l=1}^{m} \mathbf{\Gamma} -$$

$$- m(\mu \mathbf{a}_0 \mathbf{a}_0^* + \mathbf{P}\boldsymbol{\lambda}\mathbf{a}_0^* + \mathbf{a}_0 \mathbf{v}^* \mathbf{P} + \mathbf{P}\mathbf{\Gamma}) = \mathbf{O}$$

can be obtained, that is

$$\left(\sum_{l=1}^{m} \mathbf{A}^l - m\mathbf{P}\right) \boldsymbol{\lambda}\mathbf{a}_0 + \mathbf{a}_0 \mathbf{v}^* \left(\sum_{l=1}^{m} \mathbf{A}^l - m\mathbf{P}\right) + m(\mathbf{\Gamma} - \mathbf{P}\mathbf{\Gamma}) = \mathbf{O}.$$

Since the first and second terms on the left side are equal to the zero matrix, $(\mathbf{E} - \mathbf{P})\mathbf{\Gamma} = \mathbf{O}$. From this one can get

$$\mathbf{\Gamma} = \bar{\gamma} \mathbf{a}_0 \mathbf{a}_0^*.$$

2. The proof of the sufficiency of the condition can take place similarly to the total symmetric and symmetric cases.

Remark 6. Theorems 3, 4 and 5 are in connection with the testing of statistical hypotheses.

According to Theorem 3 the null hypothesis $H_{\lambda 0}$ that the column-vector $\boldsymbol{\lambda}$ of row-effects has equal components is equivalent to the null hypothesis $H'_{\lambda 0}$ according to which the expectation of the matrix of the discrepancies between the rows equals the zero matrix. Theorem 4 gives a null hypothesis H'_{v0} equivalent to the null hypothesis H_{v0} concerning the equality of the components of the column-vector $\mathbf{v}$ of column-effects $(H'_{v0}\colon M(\boldsymbol{\eta}_1 - \boldsymbol{\zeta}) = \mathbf{O})$. Theorem 5 states that for a total symmetric, cyclic and symmetric Latin square the null hypothesis H_{Γ_0}, according to which the matrix $\mathbf{\Gamma}$ of the treatments effect consists of equal elements, is equivalent to the null hypothesis H'_{Γ_0} that the expectation of the matrix of the discrepancies between treatments is the zero matrix.

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