On the existence of fixpoints

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1. *Introduction.*

Let S be the complex sphere and f a function defined in a domain D on S. If $S-D$, contains an isolated point, we assume without loss of generality that it is an essential singularity of f.

If $z\in D$ is such that $f(z)\in D$, then $f_2(z)=f((f(z))$ exists. If further $f_2(z)\in D$, then f can be iterated once more. Define $f_n(z)$ by $f_0(z)=z$, and $f_{n+1}(z)=f(f_n(z))$. We say that a point $z\in D$, is of *order* N if there exists a number N such that $f_n(z)\in D$ for $0<n\leqq N$, but $f_{N+1}(z)\notin D$. If no such N exists then we say that the point z is of *infinite order*. Let $D_{\inf}$ denote the subset of D of points of infinite order. We now say of the function f that

i) f belongs to Class I if $S-D_{\inf}$ is empty,
ii) f belongs to Class II if $S-D_{\inf}$ contains one point,
iii) f belongs to Class III if $S-D_{\inf}$ contains two points.

The case when $S-D_{\inf}$ contains more than two points is of no interest to us.

RÅDSTRÖM (5, p. 87) has proved:

a) If f belongs to class I, f has to be a rational function,
b) If f belongs to class II, f has to be an entire transcendental function,
c) If f belongs to class III, the f can be expressed in one of the two following forms

III A. $f(z)=z^{-n}\exp F(z)$, n a positive integer, F nonconstant entire,

or

III B. $f(z)=z^{n}\exp\left[F(z)+G\left(\frac{1}{z}\right)\right]$, n a positive integer, F, G non-constant entire.

If f belongs to one of the classes I, II or III, then its iterates f_n, $n=2, 3, \ldots$ belong to the same class, except that when f is of class III A, then f_n $(n>1)$ belong to class III B.

Definition 1. If $w=f_n(z)$, w is called a *successor* of z and Z is called a *predecessor* of w, in each case of *order* n.

Definition 2. If $f_n(\alpha)=\alpha$, then α is called *fixpoint of order* n. Further, if $f_k(\alpha)\neq\alpha$ for $k<n$, then α is called a fixpoint of *exact order* n. In this case the derivative $f_n'(\alpha)=\prod_{k=1}^{n-1} f'(f_k(\alpha))$ is called the *multiplier* of α.

The successor of a fixpoint α of exact order n is again a fixpoint of exact order n. The set $\{\alpha, f(\alpha), \ldots, f_{n-1}(\alpha)\}$ is called a *cycle* of order n.

Definition 3. A fixpoint α (or a cycle) of order n is called *attractive, indifferent,* or *repulsive,* according as $|f_n'(\alpha)|<1$, $=1$ or >1 respectively. If $f_n'(\alpha)=e^{2\pi\frac{q}{p}}$, where p and q are integers, then α (and with it the cycle) is called *rationally indifferent.*

The main object of study in global iteration theory are the set $\mathscr{F}=\mathscr{F}(f)$ of points in whose neighbourhood the sequence $\{f_n\}$ is not normal in the sense of Montel and the way the complement $C(\mathscr{F})$ of $\mathscr{F}$ splits into components (domains of normality).

Definition 4. The *immediate domain of attraction* D_α of a first-order attractive fixpoint α is the maximal domain of normality of $\{f_n\}$ that contains α. In D_α, $\lim_{n\to\infty} f_n(z)=\alpha$.

It is clear from the definition that D_α is a domain whose boundary belongs to $\mathscr{F}$.

In [3, p. 81], Fatou proved that if $f(z)$ belongs to class I and α is a first-order attractive fixpoint of $f(z)$, then there exists a *first-order* nonattractive fixpoint on the frontier ∂D_α of D_α. In [1], we showed that this result cannot in general be carried over to functions in class II. However, we proved that Fatou's result still holds for functions in class II, if D_α is bounded.

In this paper we prove stronger statements about functions in classes II and III.

Theorem 1. *Let* $f(z)$ *belong to the class* II *or* III. *Let* α *be a first-order attractive fixpoint of* $f(z)$ *such that*

(i) D_α *is bounded in case* $f(z)$ *belongs to class* II,

(ii) D_α *is bounded away from* 0 *and* ∞ *in case* $f(z)$ *belongs to class* III.

Then, for every $n\geqq 1$ $(n\neq 2$, *in case* (ii)$)$, *there exist cycles of fixpoints of exact order* n *on the frontier* ∂D_α *of* D_α.

2. *Preliminaries*

Definition 5. *A* set A is said to be *completely invariant* with respect to the iteration of $f(z)$ if $f(z)$ belongs to A *if and only if* z belongs zo A.

If $f(z)$ belongs to class I or II, then $\mathscr{F}(f)$ and its complement are completely invariant in this sense [3, pp. 33—41]. The same is easily seen to be true for functions in class III. It follows that if D_α is the immediate domain of attraction of the first-order attractive fixpoint α of an $f(z)$ in class I, II, or III, then $f(z)$ maps D_α into itself and $f(z)\to\partial D_\alpha$ as $z\to\partial D_\alpha$.

We know the following theorem [6, p. 131].

Theorem. [Wolff and Denjoy]. *If the regular function* $g(z)$ *maps the disk* U: $|z|<1$ *into itself and is not a bilinear transformation of* U *into itself, then the sequence* $\{g_n\}$ *has in* U *a constant limit function (which belongs to* $\overline{U}$*).* (The bar denotes closure.)

By means of a conformal mapping, we get the following obvious corollary to this theorem.

Corollary 1. *If D is a simply connected domain and the regular function $h(z)$ maps D into itself and is not a univalent map of D into itself, then every limit function of $\{h_n(z)\}$ in D is a constant.*

We also need

Lemma 1. *Under the conditions of Theorem* 1, *D_α is simply connected in both cases.*

PROOF. If λ is a simple closed curve lying in D_α and if $f(z)$ is regular in the interior of λ, then all $f_n(z)$ are regular inside and on λ, and $f_n(z)\to\alpha$ on λ, which therefore belongs to D_α. This shows that if $f(z)$ is in class II, any immediate domain of attraction D_α is simply connected, while if $f(z)$ belongs to class III, the interior of any closed $\lambda\subset D_\alpha$ also belongs to D_α so long as λ does not wind around the origin.

It remains to show that if $f(z)$ belongs to class III, then no simple closed curve λ in D_α can wind around 0. Suppose there does exist such a simple closed curve λ in D_α which winds around 0. Consider those branches of $z=f_{-1}(w)$ which may be obtained by local inversion of the power series $w=f(z)=\alpha+f'(\alpha)(z-\alpha)+\dots$, near α and continuation throughout D_α in all possible ways. Since D_α is bounded away from 0 and ∞ we see that (a) no transcendental singularities of $f_{-1}(w)$ are encountered under this continuation, (b) at most a finite number of branches of $f_{-1}(w)$ are obtained, all continuable into one another in D_α and (c) all take values in D_α for $w\in D_\alpha$. Now we can suppose λ (altered slightly, if necessary) to contain none of the singularities of these branches of $f_{-1}(w)$. Continuing the branches from a fixed $w_0\in\lambda$, around λ a finite number of times we see that there is a branch $\tilde f_{-1}(w_0)=z_0$ and an integer p such*) that continuation p times around λ brings z_0 back to z_0 for the first time. Then the image λ' of λ by $z=f_{-1}(w)$ under this continuation is a simple closed curve, since there are no singularities of f_{-1} on λ and so λ' cannot cross itself. Also by (c) $\lambda'\subset D_\alpha$.

Since $f(\lambda')$ is $p\cdot\lambda$ we see that if λ' does not wind around 0, then f is regular inside λ and f has $|p|$ zeros or poles inside λ', which is impossible. Thus λ' winds around 0. Now if λ',λ'' are any two simple curves not passing through 0 but winding around it once in the positive direction then $f(\lambda')$ and $f(\lambda'')$ have the same winding number with respect to 0, viz. $\dfrac{1}{2\pi i}\displaystyle\int\frac{f'(z)}{f(z)}\,dz=p$. Then $f_2(\lambda')=$ $=f(p\lambda)$ has the same winding number with respect to 0 as $f(p\lambda')=p^2$ and $f_n(\lambda')$ has winding number $p^n\neq 0$. But since $\lambda'\subset D_\alpha$ and λ' is compact we see that $f_n(z)\to$ $\to\alpha(\neq 0,\infty)$ uniformly on λ' and this is impossible, if $f_n(\lambda')$ winds around 0. This contradiction shows that in fact no curve such as λ can exist, and the lemma is established.

Lemma 2. *Let $f(z)$ satisfy the conditions of Theorem* 1. *Let $z=\varphi(w)$ map $U=$ $=\{w\colon |w|<1\}$ one to one onto D_α, such that $\varphi(0)=\alpha$. Then the compound*

*) p will be positive if we continue around λ in the positive direction and it will be negative if we continue in the negative direction of λ.

map $F(w)=\varphi_{-1}\circ f\circ\varphi(w)$ *of* $U\to U$ *is rational and has repulsive fixpoints of every order* n $(n\geqq 1)$ *on* $K=\partial U$.

PROOF. By Lemma 1 we see that D_α is simply connected in each case. Hence a map $z=\varphi(w)$ such as that of the hypothesis exists by Riemann's mapping theorem.

Also D_α is bounded and is a domain of regularity of $f(z)$ in class III.

Now $F(w)=\varphi_{-1}\circ f\circ\varphi(w)$ keeps the unit disc U invariant and the origin fixed. Further, since $F'(0)=f'(\alpha)$, it is clear that $w=0$ is an attractive fixpoint of $F(w)$. Hence U belongs to the immediate domain of attraction of $w=0$.

We observe that

(1) $|F(w)|\to 1$ as $|w|\to 1$, and

(2) $F(w)$ is meromorphic (indeed regular) in $|w|<1$.

Hence we can continue $f(w)$ across $|w|=1$, analytically using the reflexion principle [see 4, pp. 89—90]. By reflexion in $|w|=1$, we see that

$$F(w), \quad |w|\leqq 1$$

and,

$$\frac{1}{\overline{F\left(\frac{1}{\overline{w}}\right)}}, \quad |w|>1$$

define a complete analytic function, which is meromorphic in the closed plane. The only singularities are the poles in $|w|>1$, which are the reflexions in $|w|=1$ of the zeros of $F(w)$ in U. In particular there is a pole at infinity. Thus $F(w)$ is a regular function except for finitely many poles and hence a rational function.

Since $F(0)=0$, $|F'(0)|<1$, it is clear that $F(w)$ is not a univalent map of U onto itself. Hence $F(w)$ is at least a p-valent function in D_α, where $p\geqq 2$. Since $w=0$ is an attractive fixpoint, we see that $\lim_{n\to\infty} F_n(w)=0$ in U. Also $\lim_{n\to\infty} F_n(w)=\infty$ in $|w|>1$.

Thus, we see that on K: $|z|=1$, the sequence $\{F_n(z)\}$ is not normal. Because if $\{F_\alpha(w)\}$ is normal at any point $u\in K$, then in any neighbourhood N of u, there must be a sequence of $\{F_n\}$ converging locally uniformly to a unique limit in N. But this is impossible since,

$$\lim_{n\to\infty} F_n(w)=0 \quad \text{in} \quad N\cap U$$

and

$$\lim_{n\to\infty} F_n(w)=\infty \quad \text{in} \quad N\cap\{|w|>1\}.$$

Thus we see that $K\subset\mathscr{F}(F)$. In fact it is easy to see that $K\equiv\mathscr{F}(F)$.

We now show that $F(w)$ has repulsive fixpoints of every order n.

If there is a fixpoint ξ of $F_k(w)$ on $K(\equiv\mathscr{F}(F))$, we know from the fact that $\mathscr{F}$ is invariant under $w\to F_k(w)$ [since $\mathscr{F}$ is completely invariant] that the multiplier $F_k'(\xi)$ must be real and non-zero. It is further true that $F_k'(\xi)$ is different from ± 1, for the local theory of iteration [2, pp. 191—195] shows that if $F_k'(\xi)=$ $=\pm 1$, then in a region D_ξ of which ξ is a boundary point, we have $F_n(w)\to$ $\to\xi$ $(n\to\infty)$. But D_ξ meets either U or $w\cdot\{|w|>1\}$ where $F_n(w)\to 0$ and ∞

[as $n \to \infty$] respectively. Thus $F_k'(\xi) = \pm 1$ is impossible. Since attractive fixpoints necessarily belongs to $C(\mathscr{F})$ we know that $F_k'(\xi)$ is real and $|F_k'(\xi)| > 1$. Since $F(w)$ maps U onto itself we have $F_k'(\xi) > 1$.

We have already seen that the degree d of $F(w)$ is $\geqq 2$ so that $F(w)$ has $d+1 \geqq 3$ fixpoints of order 1 which are all different. This is because a multiple root of $F(w)-w=0$ has $F'(w)=1$. We know that 0 and ∞ are attractive, while all other fixpoints belong to K, where $F_n(w)$ does not tend to any unique limit as $n \to \infty$. Hence the $d-1$ first order fixpoints, other than 0 and ∞ are all repulsive. For any integer $k > 1$ we see that $F_k(w)$ is order d^k and has d^k+1 fixpoints, all different by the same argument as for $F(w)$. The number of fixpoints of exact order $j \geqq 2$ of $F_k(w)$ is at most $d^j+1-(d+1)$ [since fixpoints of $F(w)$ are also fixpoints of $F_j(w)$] and so the number of fixpoints of exact order less than k is at most

$$d+1+(d^2-d)+\ldots+(d^{k+1}-d) < d^k,$$

so that some of the fixpoints of $F_k(w)$ must be of exact order k. By a similar argument as used before we see that these fixpoints must belong to K.

This completes the proof of the lemma.

3. *Proof of Theorem 1.*

Let $z=\varphi(w)$ map $U=\{w: |w|<1\}$ conformally one to one onto D_α such that $\varphi(0)=\alpha$. Then by Lemma 2, the function

$$F(w) = \varphi_{-1} \circ f \circ \varphi(w)$$

is rational and has repulsive fixpoints of exact order n $(n \geqq 1)$ on $K=\partial U$.

Let γ be a repulsive fixpoint of $F(w)$ of exact order p on K, where $p \geqq 1$. This means that there is a cycle of fixpoints, say γ_i $(i=0, 1, 2, \ldots, p-1)$, $\gamma_0=\gamma_p=\gamma$.

Then $F_p'(\gamma_i)=\varrho>1$ and $F_p'(\gamma_i) \neq 0$ for each $i=1, 2, \ldots, p$. Therefore we can find branches $G_i(u)$ $(i=1, 2, \ldots, p)$ of the inverse function of $u=F(w)$, regular at γ_i and with $G_i(\gamma_i)=\gamma_{i-1}$. Furthermore $H(u)$, the inverse of $u=F_p(w)$ which maps γ_0 to γ_0, satisfies

$$H(u) = G_1 \circ \circ \circ G_{p-1}(u).$$

If we take $\varepsilon > 0$ small enough then,

$$\Omega_0 = U \cap \{|w-\gamma| < \varepsilon\},$$

$$\Omega_{i-1} = G_i \circ G_{i+1} \circ \circ \circ G_p(\Omega_0); \quad i = 1, \ldots, p$$

are all disjoint and do not meet 0, and G_{i-1} is regular and univalent in Ω_{i-1}. Moreover for small $\varepsilon (>0)$

$$G_1(\Omega_1) = G_1 \circ \circ \circ G_p(\Omega_0) = H(\Omega_0)$$

and is strictly contained in Ω_0, since $H(\gamma_0)=\gamma_0$, $H'(\gamma_0)=\dfrac{1}{F_p'(\gamma_0)}=\dfrac{1}{\varrho}<1.$ Also $H_n(u) \to \gamma_0$ in a neighborhood of γ_0 which includes Ω_0.

Define

$$\Omega_{0,n} = H_n(\Omega_0).$$

Then,

$$\bigcap(\bar{\Omega}_{0,n}) = \gamma_0,$$

$$\Omega_{0,n+1} = H(\Omega_{0,n}) \text{ which is a strict subset of } \Omega_{0,n}.$$

Define

$$\Omega_{i-1,n} = G_i \circ \circ \circ G_p(\Omega_{0,n}) \subset \Omega_{i-1}$$

so that

$$\bigcup_{l=1}^{p} (\bar{\Omega}_{i-1,n}) = \gamma_{i-1},$$

and

$$\Omega_{i-1,n+1} \subset \Omega_{i-1,n}.$$

Now corresponding to

$$\Omega_{0,n+1} = H(\Omega_{0,n}) \subset \Omega_{0,n}$$

we have regions $\Delta_{0,n} = \varphi(\Omega_{0,n}) \subset D_\alpha$, which do not contain α and are such that $\Delta_{0,n}$ there is a regular branch g of $f_{-p}\,(=\varphi \circ H \circ \varphi_{-1})$ with

$$g(\Delta_{0,n}) = \Delta_{0,n+1} \subset \Delta_{0,n}.$$

The sequence of itereats $g_m(z)$ is then normal in $\Delta_{0,n}$ for any fixed n, and since $\Delta_{0,n+1}$ is strictly contained in $\Delta_{0,n}$, we see by corollary 1, that any convergent subsequence of $\{g_m(z)\}$ is a constant, say β_0.

We show that this constant β_0 is a fixpoint of exact order p and lies on ∂D_α.

Let

(1) $$g_{m_k}(z) \to \beta_0 \quad (k \to \infty), \quad z \in \Delta_{0,n}.$$

Now, g maps $\Delta_{0,n}$ univalently onto $\Delta_{0,n+1}$ and its inverse f_p maps $\Delta_{0,n+1}$ univalently onto $\Delta_{0,n}$. Hence, if $z \in \Delta_{0,n+1}$, then $f_p(z) \in \Delta_{0,n}$ and so by (1)

$$g_{m_k}(f_p(z)) \to \beta_0.$$

But $g(f_p(z)) = z$ so that

$$f_p\{g_{m_k}(z)\} = g_{m_k-1}\{g(f_p(z))\} = g_{m_k}(f_p(z)) \to \beta_0.$$

Since $z \in \Delta_{0,n+1} \subset \Delta_{0,n}$ we know by (1) that the left hand side has limit $f_p(\beta_0)$. Thus

(2) $$f_p(\beta_0) = \beta_0.$$

Further $\beta_0 \in \bar{D}_\alpha$ since $\Delta_{0,n} \subset D_\alpha$. Also since D_α is the immediate domain of attraction of α, D_α contains no fixpoints of $f(z)$ other than α and hence $\beta_0 \in \partial D_\alpha$.

Similarly to each γ_i there corresponds a β_i, each β_i being a fixpoint of $f(z)$ of order n for some n.

We next show that β_0 is accessible from within $\Delta_{0,n} \subset D_\alpha$. We have $\Delta_{0,n+1} = g(\Delta_{0,n}) \subset \Delta_{0,n}$ and g is regular in $\Delta_{0,n}$. Take any $z_0 \in \Delta_{0,n}$ and $z_1 = g(z_0)$ in $\Delta_{0,n+1}$. Join z_0, z_1 by a path l_1 in $\Delta_{0,n}$. Surround $z_0 z_1$ by a subdomain δ_1 of $\Delta_{0,n}$. Then form $\delta_2 = g(\delta_1), \ldots, \delta_n = g(\delta_{n-1})$; $l_2 = g(l_1), \ldots, l_n = g(l_{n-1})$. Each l_n or δ_n meets l_{n+1} or δ_{n+1} respectively.

We now define the superior limit L of the sets δ_n as follows [see e.g. 7., p. 10]

$$L = \limsup \delta_n = \{t \mid \text{there exists a sequence of integers } N_q$$

$$\text{and points } z_q \in \delta_{N_q} \text{ such that } z_q \to t, \text{ as } N_q \to \infty\}.$$

Then $L \subset \partial D_\alpha$, because each t is a limit function of a sequence g_{m_k} and we have already shown that this must lie on ∂D_α. Furthermore for the same reason by (2), any $t \in L$ is a fixpoint of order p of $f(z)$ by the argument above. It follows from the definition that L is closed. Further it must be bounded, since D_α (which contains L) is bounded. Hence L is compact. We next show that L is connected.

Suppose L is not connected. Then there exist closed non-empty sets L_1 and L_2 such that $L \subset L_1 \cup L_2$, and $L_1 \cap L_2 = \emptyset$. The distance between L_1 and L_2 (which must be positive) is

$$\zeta(L_1 L_2) = 4\eta \quad \text{(say) where} \quad \eta > 0.$$

Now for some n_0, δ_n lies in an η-neighborhood of L for all $n \geqq n_0$. Otherwise there exist $n_1, n_2, \ldots,$ tending to infinity, $\xi_{n_1} \in \delta_{n_1}$, $\xi_{n_2} \in \delta_{n_2}$; $\zeta(\xi_{n_i}, L) \geqq \eta$ and ξ_{n_i} will have a point of accumulation (since they are inside the compact set $\bar{D}_\alpha$) t and $t \notin L$. This is against the definition of L.

Take $\xi_1 \in L_1$ and $\xi_2 \in L_2$. Then there exist a δ_{n_1}, $n_1 > n_0$, which meets a η-neighborhood of ξ_1 and similarly there exists a δ_{n_2}, $n_2 > n_1$ which meets a η-neighborhood of ξ_2.

Consider the chain

$$C = \delta_{n_1} \cup \delta_{n_1+1} \cup \ldots \cup \delta_{n_2},$$

where δ_{n_1} meets an η-neighborhood of ξ_1 and δ_{n_2} meets an η-neighborhood of ξ_2. Now C must be connected since $\delta_n \cap \delta_{n+1} = \emptyset$. Also C lies in an η-neighborhood of L, i.e. of L_1 and L_2.

Given $\mu_1 \in \delta_{n_1} \subset \tilde{L}_1 = \eta$-neighborhood of L_1 and $\mu_2 \in \delta_{n_2} \subset \tilde{L}_2 = \eta$-neighborhood of L_2, there exists a polygon of sides less than η with vertices at

$$\mu_1 = \mu_1^1, \mu_1^2, \ldots, \mu_1^n = \mu_2 \quad \text{all lying in } C.$$

Let μ_1^j be the last μ_1 in $\tilde{L}_1$. Then $j \neq n$ since $\mu_2 = \mu_1^n$ lies in $\tilde{L}_2$. Also $\mu_1^{j+1} \in \tilde{L}_2$. Now $\zeta(\mu_1^j, \mu_1^{j+1}) < \eta$, so that

$$\zeta(L_1 L_2) < \zeta(L_1, \mu_1^j) + \zeta(\mu_1^j, \mu_1^{j+1}, \mu_2) < \eta + \eta + \eta = 3\eta$$

which is a contradiction. We have now proved that L is connected. We have already shown it to be compact. Hence L is a continuum or else it must reduce to a single point.

We now show that L is in fact a single point. If not suppose L is a continuum. This means that we have a continuum of fixpoints of $f(z)$ or order p, lying in a finite part of the plane. This is impossible unless $f_p(z) = z$. Hence L must reduce to a single point and has β_0 as its unique element, accessible along the path $\Gamma = l_1 + l_2 + \ldots + l_n + \ldots$, in $\Delta_{0,n}$. Since $l_2 \subset \Delta_{0,n+1}$, $l_j \subset \Delta_{0,n+1-j}$, we see that Γ corresponds under φ_{-1} to a path Γ' lying ultimately inside $\Omega_{0,n+j-1}$, and so approaching $\gamma_0 = \gamma$.

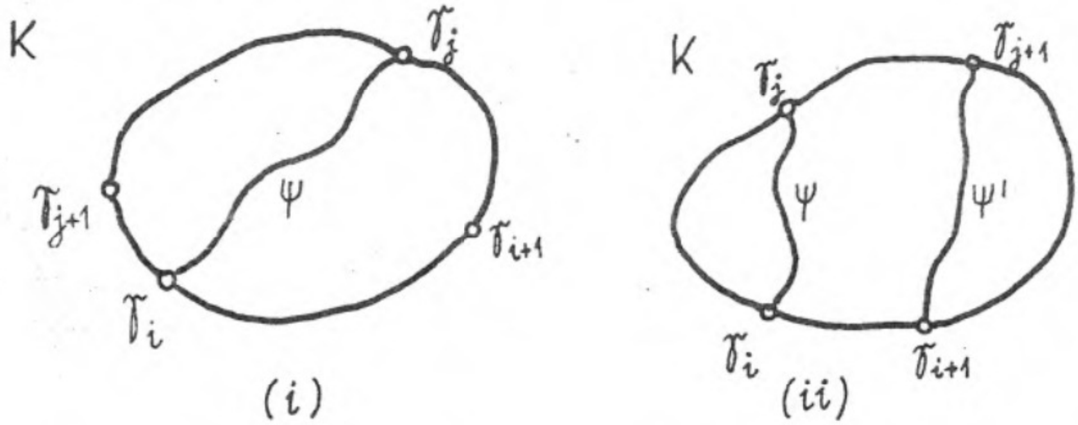
It is then clear that $\beta_1 = f(\beta_0)$ is a boundary point of ∂D_α accessible by the path $f(\Gamma)$ and corresponding under φ to γ_1 approached by $F(\Gamma')$. Similarly for $\beta_i=f_i(\beta_0)$, $i=0, 1, \ldots, p-1$ which forms a cycle of fixpoints of exact order p on ∂D_α, provided we can show that,

$$\beta_i \neq \beta_j \quad \text{for} \quad i \neq j, \quad 0 \leqq i, \quad j \leqq p-1.$$

Suppose there are i and j, $i \neq j$, such that $\beta_i=\beta_j=b$. Then β_i is accessible from some $\Delta_{i,n}$ along a path Γ_i and β_j accessible from some $\Delta_{j,n}$ along a path Γ_j. These paths are disjoint except at b. Further they correspond to two disjoint end cuts Γ_i', Γ_j' in U, ending in γ_i, γ_j respectively. We may join the ends of Γ_i', Γ_j' lying in U by an arc Γ'' to obtain a cross-cut $\Lambda=\Gamma_i'+\Gamma''+\Gamma_j'$ of U and this will correspond to $\Sigma=\varphi(\Lambda)$, a Jordan curve which lies in D_α except for the end point b. The interior of Σ will contain $\varphi(U_1)$, where U_1 is one of the regions into which Λ divides U. Boundary points of $\varphi(U_1)$ corresponding to points of K belong to $\mathscr{F}$, so Σ contains points of $\mathscr{F}$ in its interior.

For functions in class II, $\Sigma-(\beta_i)\subset D_\alpha$ so that $f_n(z)\in D_\alpha$ for $z\in\Sigma-(\beta_i)$, while $f_n(\beta_i)$ belongs to $\{\beta_1, \beta_2, \ldots, \beta_p\}$. Thus for $z\in\Sigma$, $\{f_n(z)\}$ is bounded. Since $f_n(z)$ is regular within Σ, we see that $\{f_\alpha(z)\}$ is uniformly bounded and hence normal within Σ. But this contradicts the conclusion of the previous paragraph and completes the proof of the theorem for function in class II.

For function in class III, the above argument breaks down only if Σ contains 0 in its interior. In this case observe that $\Sigma-(\beta_i)$ belongs to D_α, while 0 and ∞ are in the (connected) complement of D_α. Thus the complement of D_α meets every curve which winds around 0. We prove our theorem only for $p\geqq 3$. [The following argument breaks down if $p=2$.] Then either there are two different pairs $\beta_i=\beta_j$ *or*, there are three β's equal say $\beta_i=\beta_j=\beta_k$. In the former case we have either (i) or (ii) of the following figure, according as (γ_i, γ_j) are separated by $(\gamma_{i+1}, \gamma_{j+1})$ or not.



In (i) ψ separates $\gamma_{i+1}, \gamma_{j+1}$ and so Σ. But this is impossible. β_{i+1}, β_{j+1} are in the boundaries of different components of $D_\alpha-\Sigma$. Since $\beta_{i+1}=\beta_{j+1}$ and $\beta_{j+1}\notin\Sigma$ we have a contradiction.

In (ii) we join $\gamma_i\gamma_j$ by ψ as before and $\gamma_{i+1}, \gamma_{j+1}$ by a cut ψ' of U disjoint from ψ. The images Σ, Σ' if ψ, ψ' in D_α are then disjoint simple Jordan curves, belonging to D_α except for β_i and β_{i+1} respectively.

The ring domain A between Σ, Σ' then can be assumed not to contain 0 or the argument for functions in class II would apply.

However, on the boundary of A, $f_n(z)$ is bounded $[\in D_\alpha \cup \{0, \ldots, p-1\}]$ and so $\{f_n(z)\}$ is regular and bounded, hence normal in A and so on a curve winding round 0 drawn in A. This is impossible.

Finally, if there are three different $\beta's$: $\beta_i=\beta_j=\beta_k$ we take q on ψ and a cut ψ'' in U joining q to γ_k. The image of ψ'' in D_α is an arc Σ'' joining q on Σ to $\beta_i=$ $=\beta_j=\beta_k$. Then one of the regions bounded by Σ'' and an arc of Σ will not contain 0 and we see that the sequence $f_n(z)$ are regular and bounded in this region and hence normal. The region must however contain points of $\mathscr{F}$ corresponding to certain points of K. This gives a contradiction.

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