A note on the lower radical

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All rings considered will be associative. Standard ring and radical theory sources such as [2] and the references listed contain the definitions of those concepts not defined in the paper.

In [3] VAN LEEUWEN and HEYMAN investigated almost nilpotent rings. A ring A is called almost nilpotent if every non-zero ideal of A strictly contains a power of A. In [4] WIEGANDT generalized the statement that an almost nilpotent ring is either Baer radical or Baer semisimple and gave characterizations of the class of Baer (lower) radical rings. The purpose of this note is to obtain results corresponding to those in [4]. In what follows Z and β_s will denote the class of all zero rings and the upper radical class determined by the class of all prime simple rings [1], respectively. For any radical class R the class of all R-semisimple rings will be denoted by SR.

Definition. Let M be an arbitrary class of rings. A nonsimple ring A is called an h_M-ring if:

(i) $A/I\in M$ for every nonzero ideal I of A.

(ii) Every minimal ideal K of A belongs to M.

A non-zero simple ring A is an h_M-ring if and only if $A\in M$. H_M will denote the class of all h_M-rings determined by the class M.

It is obvious that if M is hereditary and homomorphically closed, then H_M is homomorphically closed and $M\subseteq H_M$. (We assume $0\in H_M$ for every H_M.)

Let R be any radical class and consider the following condition:

(Q) *Any H_M-ring is either R-radical or R-semisimple.*

We now consider a class M which is hereditary, homomorphically closed and closed under forming finite direct sums. Then we can prove the following

Theorem 1. *Let R be a radical class such that $R\cap H_M\neq\emptyset$. R satisfies condition* (Q) *if and only if $M\subseteq R$.*

PROOF. Suppose $M\subseteq R$ and let $A\in H_M$. If J is an ideal of A such that $0\neq J\neq A$, then $A/J\in M$. Hence 0 is the only proper ideal for which $A/0\cong A$

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can be R-semisimple. Therefore, either A cannot be mapped homomorphically onto a nonzero R-semisimple ring or $A \in SR$. In other words, either $A \in R$ or $A \in SR$, so that R satisfies condition (Q). Conversely, suppose R satisfies condition (Q). Assume first that R does not contain all the M-rings. Then there exists a ring $0 \neq K \in M$ such that $0 \neq S \cong K/R(K) \in SR \cap M$.

If R contains a ring $0 \neq X \in M$ then $A = X \oplus S \in M$ since M is closed under taking finite direct sums. Furthermore $0 \neq X = R(A) \neq A$ and hence condition (Q) is not fulfilled.

If $R \cap M = \emptyset$ then SR contains all M-rings. Since however $R \cap H_M \neq \emptyset$, there exists a nonzero ring $T \in R \cap H_M$. We now consider a subdirect sum decomposition of T, namely $T \cong \sum_i G_i$ where all the G_i's are subdirectly irreducible rings and $T/K_i \cong G_i$ for every i. Either every $G_i \in M$ or T itself is a subdirectly irreducible ring. In the first case $R \cap M \neq \emptyset$ which is a contradiction. In the second case, according to (ii) of the definition, the heart H of T belongs to M. $H = T$ implies $R \cap M \neq \emptyset$. If $H \neq T$ then $T/H \in M \cap R = \emptyset$ so that $T = H$ which again is a contradiction so that $R \cap M \neq \emptyset$. From this, and what was shown above, it is evident that $M \subseteqq R$. This completes the proof.

Corollary 1. *A radical class R coincides with the lower radical class LM determined by all M-rings, if and only if*

(i) *R satisfies condition* (Q).

(ii) $R \cap H_M \neq \emptyset$.

(iii) *for any radical class P satisfying properties* (i) *and* (ii) *it follows that $R \subseteqq P$.*

Remark. It is to be pointed out that Theorem 1 and Corollary 1 correspond to Theorem 1 and Corollary 1 in [4].

If in addition $Z \subseteqq M$, we can prove

Theorem 2. *A hereditary radical class $\emptyset \neq R \subset \beta_s$ satisfies condition* (Q) *if and only if $M \subseteqq R$.*

Proof. If $M \subseteqq R$ then Theorem 1 implies that R satisfies condition (Q).

Conversely suppose R satisfies condition (Q). Assume $M \nsubseteq R$. If R contains an M-ring $X \neq 0$, then as in Theorem 1, condition (Q) is not satisfied.

Assume $R \cap M = \emptyset$ and let $A \in R$. A can be decomposed into a subdirect sum $A \cong \sum_i A_i$ of subdirectly irreducible rings A_i. Let H be the heart of A_i. If $H^2 = H$ then $H \in S\beta_s \subseteqq SR$ which is impossible since R is hereditary and $A_i \in R$. Hence $H^2 = 0$ which implies $H \in Z \subseteqq M$. Hence $H \in R \cap M = \emptyset$ so that $H = 0$, a contradiction. Hence $R \cap M \neq \emptyset$ so that the theorem is proved.

Corollary 2. *A hereditary radical class R coincides with LM if and only if*

(i) *R satisfies condition* (Q),

(ii) *for any hereditary radical class P, satisfying condition* (Q) *it follows that $R \subseteqq P$.*

References

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