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On power values of binary forms over function fields

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Abstract. An inequality is given for the height to the "rational" solutions x, y, z of the equation $F(x, y) = z^m$.

1. Introduction

Let $\mathbb{K}$ be an algebraic function field and $F(X, Y) \in \mathbb{K}[X, Y]$ be a binary form of degree n having pairwise non-proportional linear factors in its splitting field. The equation

$$F(x, y) = z^m \text{ in } x, y, z \in \mathbb{K};\ m \in \mathbb{Z} \tag{1}$$

is a natural, common generalization of the superelliptic, Thue and Fermat equations. The first general theorem was obtained by SCHMIDT [S1], he gives an effective upper bound for the degree of the polynomial solutions. His result was extended by BRINDZA [B1], [B2] to the case of algebraic function fields when the unknowns are S-integers. By taking $z = 1$ or $y = 1$ we have Thue or superelliptic equations, respectively. For the related results we refer to [S2], [M1], [M2], [M3], [BM], [P] and [BPV].

The purpose of this note is to prove a bit surprisingly simple inequality which covers several previous results. Let $H_{\mathbb{K}}(\alpha)$ denote the additive height of a non-zero $\alpha \in \mathbb{K}$, that is

$$H_{\mathbb{K}}(\alpha) = \sum_v \max(0, v(\alpha)),$$

where v runs through the additive valuations of $\mathbb{K}$ with value group $\mathbb{Z}$.

Theorem. *If the degree of F is at least 5 then all the non-zero solutions of (1) in $x, y, z \in \mathbb{K}, m \in \mathbb{Z}$ satisfy*

$$H_{\mathbb{K}}\left(\frac{x}{y}\right) \leq 10 H_{\mathbb{K}}(z) + C,$$

where C is an effectively computable constant depending only on F and $\mathbb{K}$.

If $z = 1$ then the equation (1) can be written as

$$y^n = \frac{1}{F\left(\frac{x}{y}, 1\right)}$$

and the known properties of the additive height (see [M1] and [S2]) lead to a bound for $\max(H_{\mathbb{K}}(x), H_{\mathbb{K}}(y))$. Moreover, if x and y are relatively prime integral functions, that is the corresponding integral divisors represented by x and y are relatively prime, then

$$H_{\mathbb{K}}\left(\frac{x}{y}\right) \geq \max\left(H_{\mathbb{K}}(x), H_{\mathbb{K}}(y)\right)$$

and one can derive an effective upper bound to the exponent m, provided that z is not a constant.

Proof of the Theorem. In the sequel $c_1, \dots, c_7$ will denote effectively computable constants depending only on F and $\mathbb{K}$. In the splitting field $\mathbb{L}$ of F we get the factorization

$$F(X, Y) = a \prod_{i=1}^{n} (X - \alpha_i Y), a \neq 0, \alpha_i \neq \alpha_j.$$

Let S denote the set of additive valuations of $\mathbb{L}$ including all the infinite valuations such that $v(z) \cdot v(a) \neq 0$ and $v(\alpha_i - \alpha_j) \neq 0$ for some i, j. Then

$$|S| \leq 2H_{\mathbb{L}}(z) + c_1 = 2[\mathbb{L} : \mathbb{K}] H_{\mathbb{K}}(z) + c_1.$$

To show that

$$n | v\left(\frac{x - \alpha_i y}{x - \alpha_j y}\right); v \notin S$$

we suppose that there exists an i for which $v(x - \alpha_i y) \neq 0$ and $v(z) = 0$. It implies the existence of an index j with $v(x - \alpha_j y) \neq 0$ and

$$v(x - \alpha_i y) \cdot v(x - \alpha_j y) < 0.$$

Without loss of generality one can assume that

$$v(x-\alpha_i y) < 0 \text{ and } v(x-\alpha_j y) > 0.$$

Combining the Siegel-identity with the property of valuations we obtain

$$v(x-\alpha_k y) = v(x-\alpha_i y); k \neq j,$$

and it yields

$$v(x-\alpha_j y) = (n-1)\cdot|v(x-\alpha_i y)|,$$
$$v\left(\frac{x-\alpha_i y}{x-\alpha_j y}\right) = n\cdot v(x-\alpha_i y).$$

Put $u_1 = \frac{x-\alpha_1 y}{x-\alpha_2 y}, u_2 = \frac{x-\alpha_3 y}{x-\alpha_2 y}$. Let k_i^+ and k_i^- denote the cardinality of the set of the valuations $v \notin S$ satisfying $v(u_i) > 0$ and $v(u_i) < 0$, respectively, $i = 1, 2$. Applying the well-known inequality of Mason [M1] and the sum-formula we have

$$k_1^+ n \leq H_{\mathbb{L}}(u_1) < S + k_1^+ + k_1^- + k_2^+ + k_2^- + c_2,$$
$$k_1^- n \leq H_{\mathbb{L}}(u_1) < S + k_1^+ + k_1^- + k_2^+ + k_2^- + c_2,$$
$$k_2^+ n \leq H_{\mathbb{L}}(u_2) < S + k_1^+ + k_1^- + k_2^+ + k_2^- + c_3,$$
$$k_2^- n \leq H_{\mathbb{L}}(u_2) < S + k_1^+ + k_1^- + k_2^+ + k_2^- + c_3,$$
$$n(k_1^+ + k_1^- + k_2^+ + k_2^-) \leq 4S + 4(k_1^+ + k_1^- + k_2^+ + k_2^-) + c_4.$$

A simple calculation leads to

$$H_{\mathbb{L}}(u_1) \leq 5S + c_5 = 10[\mathbb{L}:\mathbb{K}]H_{\mathbb{K}}(z) + c_6$$

and

$$[\mathbb{L}:\mathbb{K}]H_{\mathbb{K}}\left(\frac{x}{y}\right) = H_{\mathbb{L}}\left(\frac{x}{y}\right) \leq 10[\mathbb{L}:\mathbb{K}]H_{\mathbb{K}}(z) + c_7.$$

and the theorem is proved.

References

[BM] B. Brindza and R. C. Mason, LeVeque's superelliptic equation over function fields, *Acta Arith.* **47** (1986), 167–173.

[BPV] B. Brindza, Á. Pintér and J. Végső, The Schinzel-Tijdeman Theorem over Function Fields, *C. R. Math. Rep. Acad. Sci. Canada* **16** (1994), 53–57.

[B1] B. Brindza, On the equation $F(x, y) = z^m$ over function fields, *Acta Math. Hungar.* **49** (1987), 267–275.

[B2] B. Brindza, Some new applications of an inequality of Mason, New advances in trancendence theory (Durham, 1986), *Cambridge University Press*, *Cambridge – New York*, 1988, pp. 82–89.

[M1] R. C. Mason, Diophantine Equations over Function Fields, *Cambridge University Press*, *Cambridge*, 1984.

[M2] R. C. Mason, On Thue's equation over function fields, *J. London Math. Soc. Ser. 2* **24** (1981), 414–426.

[M3] R. C. Mason, The hyperelliptic equation over function fields, *Proc. Camb. Philos. Soc.* **93** (1983), 219–230.

[P] Á. Pintér, Exponential diophantine equations over function fields, *Publ. Math. Debrecen* **41** (1992), 89–98.

[S1] W. M. Schmidt, Polynomial solutions of $F(x,y) = z^n$, *Queen's Papers in Pure Appl. Math.* **54** (1980), 33–65.

[S2] W. M Schmidt, Thue's equation over function fields, *J. Austral. Math. Soc. Ser. A* **25** (1978), 385–422.

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