Decomposition of curvature tensor of second kind in a birecurrent areal space

By H. D. PANDE and B. D. DWIVEDI (Gorakhpur)

1. *Introduction.* We consider the areal space $A_n^{(m)}$ (n: the dimension of the space; m: the order of the integral) in which the areal metric tensor is defined as ([2], p. 289)[1])

$$g_{ij}^{\alpha\beta}(x^h, \dot{x}_\varepsilon^h) = \left[\frac{m}{2}\frac{\partial^2\{L(x^h, \dot{x}_\varepsilon^h)\}^{\frac{2}{m}}}{\partial\dot{x}_\alpha^i\partial\dot{x}_\beta^j}\right], \tag{1.1}$$

where[2]) $L=L(x^h, \dot{x}_\varepsilon^h)$ satisfies the relation

$$\frac{\partial L}{\partial\dot{x}_\alpha^i}\dot{x}_\alpha^i = mL. \tag{1.2}$$

We have

$$g_{ij}^{\alpha\beta}\dot{x}_\alpha^i\dot{x}_\beta^j = mL^{\frac{2}{m}} \quad\text{and}\quad g_{ik}^{\alpha\gamma}\dot{x}_\alpha^i = L^{\left(\frac{2}{m}-1\right)}\frac{\partial L}{\partial\dot{x}_\gamma^k}. \tag{1.3}$$

The covariant derivative of a vector field $\dot{X}_\varepsilon^i(x^h, \dot{x}_\varepsilon^h)$ with respect to x^j is given by ([1])

$$X_{\varepsilon|j}^i = \frac{\partial X_\varepsilon^i}{\partial x^j} - \frac{\partial X_\varepsilon^i}{\partial\dot{x}_\lambda^1}\Gamma_{pj}^l\dot{x}_\lambda^p + \Gamma_{jl}^i X_\varepsilon^1, \tag{1.4}$$

where Γ_{jl}^i is a connection coefficient.

The commutation formula for a vector field $X_\varepsilon^i(x^h, \dot{x}_\varepsilon^h)$ involving the covariant derivative of the type (1.4) is given by ([1])

$$X_{\varepsilon|k|h}^i - X_{\varepsilon|h|k}^i = X_\varepsilon^j \underset{2}{K}{}_{jkh}^i - \frac{\partial X_\varepsilon^i}{\partial\dot{x}_\alpha^1}\underset{1}{K}{}_{pkh}^1\dot{x}_\alpha^p + X_{\varepsilon|j}^i T_{kh}^j, \tag{1.5}$$

where $\underset{1}{K}{}_{jkh}^i$ and $\underset{2}{K}{}_{jkh}^i$ are curvature tensors of first and second kinds respectively and

$$T_{kh}^i = \Gamma_{kh}^i - \Gamma_{hk}^i. \tag{1.6}$$

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[1]) The numbers in brackets refer to the references at the end of the paper.

[2]) Throughout this paper the Latin indices $i, j, k \ldots$ run over 1 to n, Greek indices $\alpha, \beta, \gamma \ldots$ run over 1 to m and $\dot{x}_\varepsilon^h = \frac{\partial x^h}{\partial t^\varepsilon}$.

The Bianchi identities corresponding to the curvature tensors of first and second kinds are given by ([1])

(1.7) $$\underset{2}{K}^{i}_{jkh|l}+\frac{\partial \Gamma^{i}_{kj}}{\partial \dot{x}^{p}_{\alpha}}\underset{1}{K}^{p}_{qhl}\dot{x}^{q}_{\alpha}+T^{q}_{kh}\underset{2}{K}^{i}_{jql}+\mathrm{Cycl}\,(k,l,h)=0$$

where cycl (k, l, h) denotes the sum of two sets of the three terms each obtained by replacing the indices (k, l, h) by first (l, h, k) and then by (h, k, l) in the expression on the left hand side of (1.7). The curvature tensor of second kind satisfies the recurrent properties of first and second order as

(1.8) $$\underset{2}{K}^{i}_{jkh|l}=v_l\underset{2}{K}^{i}_{jkh}$$

and

(1.9) $$\underset{2}{K}^{i}_{jkh|l|m}=a_{lm}\underset{2}{K}^{i}_{jkh},$$

where v_l and a_{lm} are recurrence vector and tensor fields. Under the decomposition

(1.10) $$\underset{2}{K}^{i}_{jkh}=\dot{x}^{i}_{\alpha}\Phi^{\alpha}_{jkh},$$

the decomposed tensor field Φ^{α}_{jkh} behaves like a recurrent tensor field:

(1.11) $$\Phi^{\alpha}_{jkh|l}=W_l\Phi^{\alpha}_{jkh},$$

where

(1.12) $$W_l=v_l-m^{-1}L^{-\frac{2}{m}}g^{\delta\alpha}_{li}\dot{x}^{l}_{\delta}\dot{x}^{i}_{\alpha|l}.$$

2. *Desomposition of* $\underset{2}{K}^{i}_{jkh}$ *in birecurrent areal space.*

Let us consider the decomposition of the curvature tensor of second kind as

(2.1.) $$\underset{2}{K}^{i}_{jkh}=\dot{x}^{i}_{\alpha}\psi^{\alpha}_{jkh},$$

where $\psi^{\alpha}_{jkh}=\psi^{\alpha}_{jkh}(x^i,\dot{x}^{i}_{\varepsilon})$ is a non-zero homogeneous tensor field of degree—1 in its directional arguments and is called as the decompostition tensor field.

Theorem 2.1. *The decomposition tensor field* ψ^{α}_{jkh} *behaves like recurrent curvature tensor field of second order:*

(2.2) $$\psi^{\alpha}_{jkh|l|m}=A_{lm}\psi^{\alpha}_{jkh},$$

where

(2.3) $$A_{lm}=a_{lm}-(\dot{x}^{i}_{\alpha|l|m}+W_m\dot{x}^{i}_{\alpha|l}+W_l\dot{x}^{i}_{\alpha|m})m^{-1}L^{-1}\frac{\partial L}{\partial \dot{x}^{i}_{\alpha}}.$$

PROOF. Differentiating (2.1) covariant with respect to x^l, x^m, successively and taking into account the relations (1.9) and (2.1), we get

(2.4) $$a_{lm}\dot{x}^{i}_{\alpha}\psi^{\alpha}_{jkh}=\dot{x}^{i}_{\alpha|l|m}\psi^{\alpha}_{jkh}+\dot{x}^{i}_{\alpha|l}\psi^{\alpha}_{jkh|m}+\dot{x}^{i}_{\alpha|m}\psi^{\alpha}_{jkh|l}+\dot{x}^{i}_{\alpha}\psi^{\alpha}_{jkh|l|m}.$$

Multiplying the relation (2.4) by $g_{li}^{\delta\alpha}\dot{x}_\delta^l$, we have

(2.5)
$$a_{lm}\, g_{li}^{\delta\alpha}\dot{x}_\delta^l \dot{x}_\alpha^i \psi_{jkh}^\alpha = \dot{x}_{\alpha|l|m}^i g_{li}^{\delta\alpha}\dot{x}_\delta^l \psi_{jkh}^\alpha + $$
$$+\dot{x}_{\alpha|l}^i \psi_{jkh|m}^\alpha g_{li}^{\delta\alpha}\dot{x}_\delta^l + \dot{x}_{\alpha|m}^i \psi_{jkh|l}^\alpha g_{l_i}^{\delta\alpha}\dot{x}_\delta^l + \dot{x}_\alpha^i \psi_{jkh|l|m}^\alpha g_{li}^{\delta\alpha}\dot{x}_\delta^l.$$

In the light of the relations (1.11) and (1.3), the relation (2.5) yields the theorem (2.1).

Corollary 2.1. *The recurrent tensor fields of curvature and decomposed tensor fields are not equal.*

PROOF. Obvious from the relations (2.2) and (2.3).

Theorem 2.2. *Under the decomposition* (2.1), *the following identity holds:*

(2.6)
$$\underset{2}{K}{}_{jk[h|l|m]}^i = \dot{x}_\alpha^i (a_{lm}\psi_{jkh}^\alpha + a_{mh}\psi_{jkl}^\alpha + a_{hl}\psi_{jkm}^\alpha).$$

PROOF. Adding the expressions obtained by cyclic change of the indices h, l, m and taking into account the relation (2.1), we have the required identity.

Theorem 2.3. *Under the decomposition* (2.1) *the relation*

(2.7)
$$\{(A_{lm}-A_{ml})_{|n}-(A_{lm}-A_{ml})W_n + W_s W_n T_{lm}^s - W_{s|n}T_{lm}^s - W_s T_{lm|n}^s\}\psi_{jkh}^\alpha +$$
$$+\dot{x}_{\alpha|n}^s(\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) = 0$$

holds provided ψ_{jkh}^α *is independent of direction.*

PROOF. Commuting (2.2) with respect to the indices l, and m, we get

(2.8)
$$(A_{lm}-A_{ml})\psi_{jkh}^\alpha = \psi_{jkh|l|m}^\alpha - \psi_{jkh|m|l}^\alpha.$$

With aid of the commutation formula (1.5), the relations (2.1) and (1.11), (2.8) becomes

(2.9)
$$(A_{lm}-A_{ml})\psi_{jkh}^\alpha = -\dot{x}_\alpha^s(\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) -$$
$$-\frac{\partial \psi_{jkh}^\alpha}{\partial x_\alpha^p}\underset{1}{K}{}_{qlm}^p \dot{x}_\alpha^q + Ws\,\psi_{jkh}^\alpha T_{lm}^s.$$

Differentiating (2.9) covariantly with respect to x^n and by virtue of (1.11), we get

(2.10)
$$(A_{lm}-A_{ml})_{|n}\psi_{jkh}^\alpha + (A_{lm}-A_{ml})W_n\psi_{jkh}^\alpha = -\dot{x}_{\alpha|n}^s(\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) -$$
$$-2W_n\dot{x}_\alpha^s(\psi_{skh}^\alpha \psi_{jlm}^\alpha + \psi_{jsh}^\alpha \psi_{klm}^\alpha + \psi_{jks}^\alpha \psi_{hlm}^\alpha) - \left(\frac{\partial \psi_{jkh}^\alpha}{\partial \dot{x}_\alpha^p}\underset{1}{K}{}_{qlm}^p \dot{x}_\alpha^q\right)_{|n} + (W_s\psi_{jkh}^\alpha T_{lm}^s)_{|n}.$$

In the light of the relation (2.9) and the condition imposed on ψ_{jkh}^α, a little simplification of the relation (2.10) yields the theorem (2.3).

Theorem 2.4. *Under the decomposition* (2.1), *if* W_l *is independent of* $\dot{x}^i_\alpha$, *the relation*

$$\{(A_{[lm|n]}-A_{[ln|m]})+(A_{[lm}w_{n]}-A_{[ln}w_{m]})-A_{[l\langle s\rangle}T^s_{mn]}\}\psi^\alpha_{jkh}= \tag{2.11}$$

$$=-\dot{x}^s_\alpha\{W_{[l}\psi^\alpha_{\langle j\rangle mn]}\psi^\alpha_{skh}+w_{[l}\psi^\alpha_{\langle k\rangle mn]}\psi^\alpha_{jsh}+W_{[l}\psi^\alpha_{\langle h\rangle mn]}\psi^\alpha_{jks}+W_s\psi^\alpha_{jks}\psi^\alpha_{[lmn]}\}+$$

$$+\frac{\partial\psi^\alpha_{jkh}}{\partial\dot{x}^p_\alpha}W_{[l}\underset{1}{K}{}^p_{\langle q\rangle mn]}\dot{x}^q_\alpha$$

holds[3]).

PROOF. In the light of (1.11) the covariant differentiation of (2.2) with respect to x^n yields

$$\psi^\alpha_{jkh|l|m|n}=(A_{lm|n}+A_{lm}w_n)\psi^\alpha_{jkh}. \tag{2.12}$$

Commuting the indices m, n in (2.12), we have

$$\psi^\alpha_{jkh}\{(A_{lm|n}-A_{ln|m})+(A_{lm}W_n-A_{ln}W_m)\}=\psi^\alpha_{jkh|l|m|n}-\psi^\alpha_{jkh|l|n|m}. \tag{2.13}$$

With the help of commutation formula (1.5) and the relations (1.11) and (2.1), (2.13) yields

$$\{(A_{lm|n}-A_{ln|m})+(A_{lm}W_n-A_{ln}w_m)\}\psi^\alpha_{jkh}= \tag{2.14}$$

$$=-W_l\psi^\alpha_{skh}\dot{x}^s_\alpha\psi^\alpha_{jmn}-W_l\psi^\alpha_{jsh}\dot{x}^s_\alpha\psi^\alpha_{kmn}-$$

$$-W_l\psi^\alpha_{jks}\dot{x}^s_\alpha\psi^\alpha_{hmn}-W_s\psi^\alpha_{jkh}\dot{x}^s_\alpha\psi^\alpha_{lmn}-\frac{\partial W_l}{\partial\dot{x}^p_\alpha}\psi^\alpha_{jkh}$$

$$\underset{1}{K}{}^p_{qmn}\dot{x}^q_\alpha-\frac{\partial\psi^\alpha_{jkh}}{\partial\dot{x}^p_\alpha}W_l\underset{1}{K}{}^p_{qmn}\dot{x}^q_\alpha+A_{ls}\psi^\alpha_{jkh}T^s_{mn}.$$

Changing the indices l, m, and n cyclically in (2.14) and adding the obtained results, we get the theorem (2.4).

Theorem 2.5. *Under the decomposition* (2.1), *if connection coefficient is independent of direction, the relation*

$$\psi^\alpha_{j[kh}a_{l]m}+\psi^\alpha_{jq[l}T^q_{kh]|m}+v_m\psi^\alpha_{jq[l}T^q_{kh]}=0 \tag{2.15}$$

holds.

PROOF. Differentiating (1.7) covariantly with respect to x^m and taking into account the condition imposed on Γ^i_{jk} and the relations (1.8), (1.9), we have

$$a_{lm}\underset{2}{K}{}^i_{jkh}+T^q_{kh|m}\underset{2}{K}{}^i_{jql}+T^q_{kh}v_m\underset{2}{K}{}^i_{jql}+a_{hm}\underset{2}{K}{}^i_{jlk}+T^q_{lk|m}\underset{2}{K}{}^i_{jqh}+ \tag{2.16}$$

$$+T^q_{lk}v_m\underset{2}{K}{}^i_{jqh}+a_{km}\underset{2}{K}{}^i_{jhl}+T^q_{hl|m}K^i_{jqk}+T^q_{hl}v_m\underset{2}{K}{}^i_{jqk}=0.$$

In the light of (2.1), a little simplification of (2.16) yields the required results.

[3]) $A_{[lm|n]}=A_{lm|n}+A_{mn|l}+A_{nl|m}$. The indices in brackets $\langle\rangle$ are from symmetric and skew symmetric parts.

References

[1] H. RUND, A geometrical theory of multiple integral problems in the calculus of variations. *Can. J. Math.* **20** (1968), 639—657.

[2] B. N. PRASAD and V. V. BOSE, Curvature tensors in Areal space, Pure and Applied Math. Sci. 4, 1—2, 1976.

[3] B. B. SINHA, Decomposition of recurrent curvature tensor fields of second order, *Progress of Mathematics* **6**, No. 1, 1972.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR (U.P.), INDIA

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