Powers of the augmentation ideal in the Witt ring of a field

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Let F be a formally real field. Denote the Witt ring of F, its augmentation ideal, and its nil radical by $W(F)$, $I(F)$, and $W_t(F)$, respectively. Pfister's local-global principle says that $W_t(F)$ is the kernel of the total signature map on $W(F)$ [7, Satz 22].

Theorem. *Let s be a positive integer. Suppose F has only finitely many distinct places into the real numbers. Then $I(F)^s+W_t(F)$ is exactly the set of elements of $W(F)$ whose signatures at every ordering of F are divisible by 2^s.*

This theorem gives an affirmative answer to Lam's "Open Problem B" [5, p. 49] in the case of fields admitting only a finite number of real-valued places.

I believe that in a narrow sense the proof given here of the theorem is the first. However, M. MARSHALL has recently observed (see [6, footnote on p. 611]) that the theorem follows fairly directly from work he had done a few years ago [6, Theorem 7]. Even more recently, TOM CRAVEN discovered a third proof [3]. The proof presented here, which is quite different from that of Marshall or Craven, is an application of the arithmetic structure theory in [2, § 6] for reduced Witt rings of fields with only finitely many real-valued places. We hope interest in such a proof is increased by the recent result of BECKER and BRÖCKER generalizing this arithmetic structure theory to arbitrary fields [1, Theorem 6.5]. The methods of [2] will be used to reduce the proof of the above theorem to that of the following lemma on subdirect products of integral group rings.

Lemma. *Suppose $n\geqq 2$. Let $\Lambda_1, \dots, \Lambda_n$ and $\Delta_2, \dots, \Delta_n$ be groups of exponent two and let $u_{i-1}\colon \Lambda_{i-1}\to\Delta_i$ and $v_i\colon \Lambda_i\to\Delta_i$ be surjective group homomorphisms, for $i=2, 3, \dots, n$. For each i, $1\leqq i\leqq n$, let I_i be the kernel of the ring homomorphism from the group ring $Z(\Lambda_i)$ to $Z_2=Z/2Z$ which maps each element of Λ_i to $1+2Z$. Define $\Theta=\Theta_n\colon \Pi_{i=1}^n Z(\Lambda_i)\to\Pi_{i=2}^n Z_2(\Delta_i)$ by the formula*

$$\Theta\big((\lambda_i)_{1\leqq i\leqq n}\big)=\big(\bar{u}_{i-1}(\lambda_{i-1})+\bar{v}_i(\lambda_i)\big)_{2\leqq i\leqq n}.$$

(For notation, see below.) *For each $s\geqq 0$ let $J_s=\operatorname{Ker}\Theta\cap\prod_{i=1}^n I_i^s$. Then for each $s\geqq 0$, $\operatorname{Ker}\Theta$ is a commutative unitary ring, J_s is an ideal of $\operatorname{Ker}\Theta$, and $J_s=J_1^s$.*

[1]) Partially supported by National Science Foundation grant no. MCS77—04024.

In the above lemma $\bar{u}_{i-1}$ and $\bar{v}_i$ are the compositions of the homomorphisms $u_{i-1}^*\colon Z(\Lambda_{i-1})\to Z(\Delta_i)$ and $v_i^*\colon Z(\Lambda_i)\to Z(\Delta_i)$ induced by u_{i-1} and v_i with the canonical map $Z(\Delta_i)\to Z_2(\Delta_i)$. We will write u_{i-1} for u_{i-1}^* and v_i for v_i^*.

We now prove the lemma. It is easy to check that $\operatorname{Ker}\Theta$ is a commutative unitary ring with ideal J_s and that $J_s \supseteq J_1^s$. It remains to prove the reverse inclusion. Let $a=(a_1, \dots, a_n)$ be in J_s. We proceed by induction on n. Pick homomorphisms $f\colon \Delta_n\to\Lambda_n$ and $g\colon \Delta_2\to\Lambda_1$ which are right inverses to the surjections v_n and u_1, respectively. First suppose that $n>2$. By induction on n one can assume that $(a_1, \dots, a_{n-1})$ is in $(\operatorname{Ker}\Theta_{n-1}\cap \Pi_{i=1}^{n-1} I_i)^s$. Θ_{n-1} denotes here the obvious analogue of $\Theta=\Theta_n$. One checks that $a'=(a_1, \dots, a_{n-1}, fu_{n-1}(a_{n-1}))$ is in J_1^s. (The map $(b_1, \dots, b_{n-1})\to(b_1, \dots, b_{n-1}, fu_{n-1}(b_{n-1}))$ is a ring homomorphism.) Similarly, we have $a''=(gv_2(a_2), a_2, \dots, a_n)$ in J_1^s. We may suppose without loss of generality that $a_i=0$ for all $i<n$ (replace a by $a-a'$). But then $a=a''$ (since $a_1=a_2=0$), which is in J_1^s. Now suppose that $n=2$. Let $b=a_2-fu_1(a_1)$ and note $\bar{v}_2(b)=0$. Hence $v_2(b)=2c$ for some c in $Z(\Delta_2)$. Note that since b is in I_2^s, we have $2g(c)$ in I_1^s, and so $g(c)$ is in I_1^{s-1}. (For the last assertion apply [4, Theorem 5.13, (7) and (8)]: $Z(\Lambda_1)$ is isomorphic to the Witt ring of some superpythagorean field.) Write

$$a = (a_1, fu_1(a_1))+(gv_2(b), b)-(2, 0)(g(c), f(c)).$$

Clearly now, a is in J_1^s; the lemma is proved.

Proof of Theorem. Let $s\geqq 0$. Let $\sigma_1, \dots, \sigma_n$ denote the distinct real-valued places on F, indexed so that

$$\sigma_i^{-1}(R)\cdot\sigma_{i-1}^{-1}(R) \subseteq \sigma_{i-1}^{-1}(R)\cdot\sigma_j^{-1}(R)$$

whenever $i\leqq j\leqq n$ (R denotes the real numbers). For each $i=1, \dots, n$, let Λ_i denote the square factor group of the value group of σ_i, and let F_i denote the ultracompletion of F at σ_i (i.e., the maximal extension of F admitting a real-valued place extending σ_i and having the same value group as σ_i [2, Lemma 2.1]). Further, for $i=2, 3, \dots, n$, let Δ_i denote the square factor group of the value group of the valuation ring $\sigma_{i-1}^{-1}(R)\cdot\sigma_i^{-1}(R)$; we have canonical homomorphisms $u_{i-1}\colon \Lambda_{i-1}\to\Delta_i$ and $v_i\colon \Lambda_i\to\Delta_i$ [2, 3.2]. Let $W_{\mathrm{red}}(F)=W(F)/W_t(F)$ and let t denote the total signature map from $W_{\mathrm{red}}(F)$ to $\Pi_X Z$ (the set of orderings of F is denoted by X). Let $I_{\mathrm{red}}(F)$ denote the image of $I(F)$ in $W_{\mathrm{red}}(F)$. We have a commutative diagram

$$\begin{array}{ccccc} W_{\mathrm{red}}(F) & \xrightarrow{t_0} & \prod_{i=1}^{n} W(F_i) & \xrightarrow{t_1} & \prod_X Z \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Phi} & & \\ \operatorname{Ker}\Theta & \xrightarrow{j} & \prod_{i=1}^{n} Z(\Lambda_i) & \xrightarrow{\Theta} & \prod_{i=2}^{n} Z_2(\Delta_i). \end{array}$$

Here, t_0 is induced by the inclusions $F\to F_i$ and t_1 by the total signature maps on the $W(F_i)$ (so, $t=t_1t_0$ [2, Lemma 3.1]). j denotes the inclusion map and Θ is from the Lemma. The map Φ is the product of isomorphisms $W(F_i)\to Z(\Lambda_i)$ obtained as follows. For each i pick an ordering P_i of F_i. Then there is a unique isomorphism from $W(F_i)$ to $Z(\Lambda_i)$ which takes each one-dimensional form $\langle a \rangle$ to the image of a in Λ_i if a is in P_i and to the negative of this image otherwise [2, Theorem 2.5]. Φ induces an isomorphism $\varphi: W'_{red}(F) \to \operatorname{Ker} \Theta$, and in fact

$$I_{red}(F)^s = \varphi^{-1}\varphi(I_{red}(F)^s) = \varphi^{-1}(J_1^s)$$

[2, Corollary 6.5]. The Lemma tells us that this equals

$$\varphi^{-1}(J_s) = \varphi^{-1} j^{-1}\left(\prod_{i=1}^{n} I_i^s\right)$$

which by the commutativity of our diagram equals

$$t_0^{-1}\Phi^{-1}\left(\prod_{i=1}^{n} I_i^s\right) = t_0^{-1}\left(\prod_{i=1}^{n} I(F_i)^s\right).$$

Since the F_i are superpythagorean we may apply [4, Theorem 5.13 (7)] to conclude that

$$t_0^{-1}\left(\prod_{i=1}^{n} I(F_i)^s\right) = t_0^{-1}t_1^{-1}(2^s \prod_X Z) = t^{-1}(2^s \prod_X Z).$$

Thus, $I_{red}(F)^s = t^{-1}(2^s \Pi_X Z)$, which was to be proved.

References

[1] E. Becker and L. Bröcker, On the description of the reduced Witt ring, *J. Algebra* **52** (1978), 328—346.

[2] R. Brown, The reduced Witt ring of a formally real field, *Trans. Amer. Math. Soc.* **230** (1977), 257—292.

[3] T. Craven, Characterizing reduced Witt rings of fields, *J. Algebra*, **53** (1978), 68—77.

[4] R. Elman and T. Y. Lam, Quadratic forms over formally real fields and Pythagorean fields, *Amer. J. Math.* **94** (1972), 1155—1194.

[5] T. Y. Lam, Ten lectures on quadratic forms over fields, *Conference on Quadratic Forms — 1976*, Queen's papers on pure and applied math., No. **46**, *Kingston, Canada*, 1977.

[6] M. Marshall, Some local-global principles for formally real fields, *Canad. J. Math.* **29** (1977), 606—614.

[7] A. Pfister, Quadratische Formen in beliebigen Körpern, *Invent. Math.* **1** (1966), 116—132.

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(Received January 10, 1978.)