The Lie derivatives in complex areal space

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Abstract. The deformation theories in FINSLER and CARTAN spaces were developed by M. S. KNEBELMAN [1], E. T. DAVIES [2], RUND [3] and YANO [7]. These theories have also been studied in an areal space by TAKANORI IGARASHI [9] and PRASAD [11]. The geometry of spaces in these works are based on real coordinate system. The purpose of this paper is to investigate Lie derivatives in the complex areal space. After giving the outlines of complex areal spaces in § 1 we define the Lie derivative of a vector field in § 2. The section 3 is devoted for rewriting the Lie derivative of a vector field in terms of covariant partial derivatives with respect to z^1, z^{1*}. In § 4 the Lie derivative of the connection coefficients have been obtained. In the last section the concept of the areal motion has been introduced.

Throughout this paper the Latin indices $i, j, h, \ldots$ run over 1 to n while Greek indices $\alpha, \beta, \ldots$ run over 1 to m.

1. *Complex areal spaces.*

We consider a $2n$ dimensional real manifold X_{2n} (of class C^∞) referred to local coordinates (x^j, y^j). Corresponding to each point P of X_{2n} we introduce complex numbers

$$z^j = x^j + iy^j \quad (i^2 = -1) \tag{1.1}$$

which may be regarded as the complex coordinate of P (with respect to given coordinate system). If there exist complex coordinate neighbourhood $U(z^j)$, $U(\bar{z}^j)$ (where $\bar{z}^j$ refer to another local coordinate system) such that in the intersection of these neighbourhoods, we have

$$\bar{z}^j = \bar{z}^j(z^h) \quad \det \left\| \frac{\partial \bar{z}^j}{\partial z^h} \right\| \neq 0, \tag{1.2}$$

where $\bar{z}^j(z^h)$ are holomorphic functions of z^h, then space X_{2n} is said to admit a complex structure. Under these circumstances X_{2n} is called a complex space of (complex) dimension n and is denoted by C_n.

With (1.1) we may associate the conjugate complex

$$z^{j*} = x^j - iy^j \tag{1.3}$$

so that (1.2) carries with it the corresponding conjugate complex transformation

$$\bar{z}^{j*} = \bar{z}^{j*}(z^{h*}). \tag{1.4}$$

An analytic m-dimensional subspace C_m of C_n $(m<n)$ is represented parametrically by the equations ([8] page 104)

(1.5) $$z^j = z^j(u^\alpha), \quad z^{j^*} = z^{j^*}(u^{\alpha^*})$$

in which z^j, z^{j^*} are holomorphic functions of the complex variables u^α, u^{α^*} respectively. Thus the derivatives $\dot{z}^j_\alpha = \dfrac{\partial z^j}{\partial u^\alpha}$ and their complex conjugate $\dot{z}^{j^*}_{\alpha^*} = \dfrac{\partial z^{j^*}}{\partial u^{\alpha^*}}$ are defined, each of which is an element of an $n\times m$ matrix which is always supposed to be of rank m.

Now we consider real Lagrange function L of the form

(1.6) $$L = L(z^j, z^{j^*}, \dot{z}^j_\alpha, \dot{z}^{j^*}_{\alpha^*})$$

satisfying the conditions

(A) The function L is of class C^4 in all its arguments and it is scalar with respect to the transformations (1.2) and (1.4).

(B) The function L is positive for all independent sets of arguments $\dot{z}^j_\alpha, \dot{z}^{j^*}_{\alpha^*}$.

(C) The integral

(1.7) $$I = \int_G L\, du^1 \wedge du^2 \wedge \ldots \wedge du^m \wedge du^{1^*} \wedge \ldots \wedge du^{m^*}$$

over a fixed region G of C_m is invariant under the holomorphic transformation of the complex parameters

(1.8) $$\bar{u}^\alpha = \bar{u}^\alpha(u^\beta), \quad \bar{u}^{\alpha^*} = \bar{u}^{\alpha^*}(u^{\beta^*}).$$

(D) The $nm\times nm$ determinant

$$D = \det \left\| m \frac{\partial^2 L^{1/m}}{\partial \dot{z}^h_\beta \partial \dot{z}^{j^*}_{\alpha^*}} \right\|$$

is non vanishing for linearly independent quantities $\dot{z}^h_\alpha, \dot{z}^{j^*}_{\alpha^*}$.

The condition C is equivalent to the relation [4]

(1.9) $$\text{(a)}\ \frac{\partial L}{\partial \dot{z}^j_\alpha} \dot{z}^j_\beta = \delta^\alpha_\beta L, \quad \text{(b)}\ \frac{\partial L}{\partial \dot{z}^{j^*}_{\alpha^*}} \dot{z}^{j^*}_{\beta^*} = \delta^{\alpha^*}_{\beta^*} L.$$

In view of (1.9)a and (1.9)b we have ([10])

(1.10) $$2mL^{1/m} = g^{\beta\alpha}_{hj} \dot{z}^h_\beta \dot{z}^j_\alpha + 2 g^{\beta\alpha^*}_{hj^*} \dot{z}^h_\beta \dot{z}^{j^*}_{\alpha^*} + g^{\beta^*\alpha^*}_{h^*j^*} \dot{z}^{h^*}_{\beta^*} \dot{z}^{j^*}_{\alpha^*},$$

where

(1.11) $$g^{\beta\alpha}_{hj}(z^l, z^{l^*}, \dot{z}^l_\lambda, \dot{z}^{l^*}_{\lambda^*}) = m \frac{\partial^2 L^{1/m}}{\partial \dot{z}^h_\beta \partial \dot{z}^j_\alpha},$$

(1.12) $$g^{\beta\alpha^*}_{hj^*}(z^l, z^{l^*}, \dot{z}^l_\lambda, \dot{z}^{l^*}_{\lambda^*}) = m \frac{\partial^2 L^{1/m}}{\partial \dot{z}^h_\beta \partial \dot{z}^{j^*}_{\alpha^*}},$$

(1.13) $$g^{\beta^*\alpha^*}_{h^*j^*}(z^l, z^{l^*}, \dot{z}^l_\lambda, \dot{z}^{l^*}_{\lambda^*}) = m \frac{\partial^2 L^{1/m}}{\partial \dot{z}^{h^*}_{\beta^*} \partial \dot{z}^{j^*}_{\alpha^*}}.$$

From (1.10) it is evident that if L is interpreted as a measure of the area dA of an element of m-dimensional complex subspace ($2m$ dimensional real subspace) spanned by $\dot{z}^j_\alpha, \dot{z}^{j*}_{\alpha*}$ at the points z^1, z^{1*} of C_n in the sense that

$$dA = L(z^j, z^{j*}, \dot{z}^j_\alpha, \dot{z}^{j*}_{\alpha*})\, du^1 \wedge \ldots \wedge du^m \wedge du^{1*} \wedge \ldots \wedge du^{m*}, \tag{1.14}$$

then the tensors (1.11), (1.12) and (1.13) can be regarded as a suitable areal metric tensor ([6] page 289). It is to be noted that $g^{\beta\alpha}_{hj}$ is symmetric in pairs of indices such as (β, h), (α, j). The similar symmetries exist for tensors $g^{\beta\alpha*}_{hj*}$ and $g^{\beta*\alpha*}_{h*j*}$. Furthermore $g^{\beta\alpha}_{hj} \neq g^{\beta\alpha}_{jh}$.

It has been proved in [10] that

$$\text{(a)}\ g^{\beta\gamma}_{hk}\dot{z}^h_\beta = 0 \quad \text{(b)}\ g^{\beta*\gamma*}_{h*k*}\dot{z}^{h*}_{\beta*} = 0. \tag{1.15}$$

In view of (1.15)a and (1.15)b, the equation (1.10) reduces to

$$mL^{1/m} = g^{\beta\alpha*}_{hj*}\dot{z}^h_\beta \dot{z}^{j*}_{\alpha*}. \tag{1.16}$$

The connection coefficients Γ^i_{kj} and Γ^{i*}_{k*j*} defined by ([10])

$$\Gamma^i_{kj} = \frac{1}{m}\left[\frac{\partial g^{im*}_{\beta\alpha*}}{\partial \dot{z}^j_\beta}\frac{\partial g^{\gamma\alpha*}_{pm*}}{\partial z^k}\dot{z}^p_\gamma + g^{im*}_{\beta\alpha*}\frac{\partial g^{\beta\alpha*}_{jm*}}{\partial z^k}\right], \tag{1.17}$$

$$\Gamma^{i*}_{k*j*} = \frac{1}{m}\left[\frac{\partial g^{i*m}_{\beta*\alpha}}{\partial \dot{z}^{j*}_{\beta*}}\frac{\partial g^{\gamma*\alpha}_{p*m}}{\partial z^{k*}}\dot{z}^{p*}_{\gamma*} + g^{i*m}_{\beta*\alpha}\frac{\partial g^{\beta*\alpha}_{j*m}}{\partial z^{k*}}\right], \tag{1.18}$$

are used to define the covariant partial derivatives of a vector field $X^i_\varepsilon(z^1, z^{1*}, \dot{z}^1_\lambda, \dot{z}^{1*}_{\lambda*})$ with respect to z^j and z^{j*}. These are given by

$$X^i_{\varepsilon|j} = \frac{\partial X^i_\varepsilon}{\partial z^j} - \frac{\partial X^i_\varepsilon}{\partial \dot{z}^l_\lambda}\Gamma^l_{mj}\dot{z}^m_\lambda + \Gamma^i_{jl}X^l_\varepsilon, \tag{1.19}$$

$$X^i_{\varepsilon|j*} = \frac{\partial X^i_\varepsilon}{\partial z^{j*}} - \frac{\partial X^i_\varepsilon}{\partial \dot{z}^{l*}_{\lambda*}}\Gamma^{l*}_{m*j*}\dot{z}^{m*}_{\lambda*}, \tag{1.20}$$

Similarly the covariant partial derivatives of the vector field $X^{i*}_{\varepsilon*}(z^1, z^{1*}, \dot{z}^1_\lambda, \dot{z}^{1*}_{\lambda*})$ is given by

$$X^i_{\varepsilon*|j} = \frac{\partial X^{i*}_{\varepsilon*}}{\partial z^j} - \frac{\partial X^{i*}_{\varepsilon*}}{\partial \dot{z}^l_\lambda}\Gamma^l_{mj}\dot{z}^m_\lambda, \tag{1.21}$$

$$X^{i*}_{\varepsilon*|j*} = \frac{\partial X^{i*}_{\varepsilon*}}{\partial z^{j*}} - \frac{\partial X^{i*}_{\varepsilon*}}{\partial \dot{z}^{l*}_{\lambda*}}\Gamma^{l*}_{m*j*}\dot{z}^{m*}_{\lambda*} + \Gamma^{i*}_{j*l*}X^{l*}_{\varepsilon*}. \tag{1.22}$$

The connection coefficients Γ^i_{kj} and Γ^{i*}_{k*j*} is not in general symmetric in k, j. Therefore, we can also define the second type of covariant partial derivatives of X^i_ε and $X^{i*}_{\varepsilon*}$ with respect to z^j and z^{j*} respectively,

$$X^i_{\varepsilon\|j} = \frac{\partial X^i_\varepsilon}{\partial z^j} - \frac{\partial X^i_\varepsilon}{\partial \dot{z}^l_\lambda}\Gamma^l_{mj}\dot{z}^m_\lambda + \Gamma^i_{lj}X^l_\varepsilon, \tag{1.23}$$

$$X^{i*}_{\varepsilon*\|j*} = \frac{\partial X^{i*}_{\varepsilon*}}{\partial z^{j*}} - \frac{\partial X^{i*}_{\varepsilon*}}{\partial \dot{z}^{l*}_{\lambda*}}\Gamma^{l*}_{m*j*}\dot{z}^{m*}_{\lambda*} + \Gamma^{i*}_{l*j*}X^{l*}_{\epsilon*}. \tag{1.24}$$

2. *Lie derivatives of a vector field.*

Let us consider an infinitesimal point transformation of the form

(2.1) (a) $$\bar{z}^i = z^i + v^i(z^h)\, dt,$$

(2.1) (b) $$\bar{z}^{i^*} = z^{i^*} + v^{i^*}(z^{h^*})\, dt,$$

where dt is an infinitesimal constant and v^i, v^{i^*} are holomorphic functions of z^h, z^{h^*}. The transformations (2.1)a and (2.1)b carries the point (z^i, z^{i^*}) of the subspace C_m

$$z^i = z^i(u^\alpha), \quad z^{i^*} = z^{i^*}(u^{\alpha^*})$$

to the neighbouring point $(\bar{z}^i, \bar{z}^{i^*})$ of the subspace $\bar{C}_m$

$$\bar{z}^i = \bar{z}^i(u^\alpha), \quad \bar{z}^{i^*} = \bar{z}^{i^*}(u^{\alpha^*}),$$

u^α, u^{α^*} being fixed and $v^i\ (z^h) = 0 = v^{i^*}\ (z^{h^*})$ give the boundary of C_m and $\bar{C}_m$.

Under these transformations the components $\dot{z}^i_\alpha$ and $\dot{z}^{i^*}_{\alpha^*}$ are deformed as

(2.2) $$\text{(a)}\ \dot{\bar{z}}^i_\alpha = \dot{z}^i_\alpha + \frac{\partial v^i}{\partial z^j}\dot{z}^j_\alpha\, dt, \quad \text{(b)}\ \dot{\bar{z}}^{i^*}_{\alpha^*} = \dot{z}^{i^*}_{\alpha^*} + \frac{\partial v^{i^*}}{\partial z^{j^*}}\dot{z}^{j^*}_{\alpha^*}\, dt,$$

where $$\dot{\bar{z}}^i_\alpha = \frac{\partial \bar{z}^i}{\partial u^\alpha}, \quad \dot{\bar{z}}^{i^*}_{\alpha^*} = \frac{\partial \bar{z}^{i^*}}{\partial u^{\alpha^*}}.$$

The variations of $z^i, z^{i^*}, \dot{z}^i_\alpha$ and $\dot{z}^{i^*}_{\alpha^*}$ under (2.1)a and (2.1)b are represented in the form

(2.3) $$\text{(a)}\ \delta z^i = \bar{z}^j - z^i = v^i\, dt \quad \text{(b)}\ \delta z^{i^*} = \bar{z}^{i^*} - z^{i^*} = v^{i^*}\, dt,$$

(2.4) $$\text{(a)}\ \delta \dot{z}^i_\alpha = \dot{\bar{z}}^i_\alpha - \dot{z}^i_\alpha = \frac{\partial v^i}{\partial z^j}\dot{z}^j_\alpha\, dt \quad \text{(b)}\ \delta \dot{z}^{i^*}_{\alpha^*} = \dot{\bar{z}}^{i^*}_{\alpha^*} - \dot{z}^{i^*}_{\alpha^*} = \frac{\partial v^{i^*}}{\partial z^{j^*}}\dot{z}^{j^*}_{\alpha^*}\, dt.$$

If a geometric object $\Omega(z^l, z^{l^*}, \dot{z}^l_\lambda, \dot{z}^{l^*}_{\lambda^*})$ is transformed to $\bar{\Omega}(\bar{z}^l, \bar{z}^{l^*}, \dot{\bar{z}}^l_\lambda, \dot{\bar{z}}^{l^*}_{\lambda^*})$ by (2.1)a and (2.1)b then

(2.5) $$\overset{v}{d}\Omega = \bar{\Omega}(\bar{z}^l, \bar{z}^{l^*}, \dot{\bar{z}}^l_\lambda, \dot{\bar{z}}^{l^*}_{\lambda^*}) - \Omega(z^l, z^{l^*}, \dot{z}^l_\lambda, \dot{z}^{l^*}_{\lambda^*}) =$$
$$= \frac{\partial \Omega}{\partial z^l}\delta z^l + \frac{\partial \Omega}{\partial z^{l^*}}\delta z^{l^*} + \frac{\partial \Omega}{\partial \dot{z}^l_\lambda}\partial \dot{z}^l_\lambda + \frac{\partial \Omega}{\partial \dot{z}^{l^*}_{\lambda^*}}\partial \dot{z}^{l^*}_{\lambda^*}.$$

On the other hand if we interpret (2.1)a and (2.1)b as an infinitesimal coordinate transformation, then neglecting higher order terms with respect to dt, we have

(2.6) $$\text{(a)}\ \frac{\partial \bar{z}^i}{\partial z^l} = \delta^i_l + \frac{\partial v^i}{\partial z^l}\, dt \quad \text{(b)}\ \frac{\partial \bar{z}^{i^*}}{\partial z^{l^*}} = \delta^{i^*}_{l^*} + \frac{\partial v^{i^*}}{\partial z^{l^*}}\, dt,$$

(2.7) $$\text{(a)}\ \frac{\partial z^l}{\partial \bar{z}^i} = \delta^l_i - \frac{\partial v^l}{\partial z^i}\, dt \quad \text{(b)}\ \frac{\partial z^{l^*}}{\partial \bar{z}^{i^*}} = \delta^{l^*}_{i^*} - \frac{\partial v^{l^*}}{\partial z^{i^*}}\, dt.$$

When the geometric object $\Omega(z^l, z^{l*}, \dot z^l_\lambda, \dot z^{l*}_{\lambda*})$ is transformed to $\Omega(\bar z^l, \bar z^{l*}, \dot{\bar z}^l_\lambda, \dot{\bar z}^{l*}_{\lambda*})$ by the coordinate transformation (2.1)a and (2.1)b then we have

$$\overset{m}{d}\Omega = \Omega(\bar z^l, \bar z^{l*}, \dot{\bar z}^l_\lambda, \dot{\bar z}^{l*}_{\lambda*}) - \Omega(z^l, z^{l*}, \dot z^l_\lambda, \dot z^{l*}_{\lambda*}). \tag{2.8}$$

The Lie derivative of Ω with respect to (v^i, v^{i*}) is defined as ([7], [3])

$$£_v \Omega = \lim_{dt\to 0} \frac{\overset{v}{d}\Omega - \overset{m}{d}\Omega}{dt} = \lim_{dt\to 0} \frac{\bar\Omega(\bar z^l, \bar z^{l*}, \dot{\bar z}^l_\lambda, \dot{\bar z}^{l*}_{\lambda*}) - \Omega(\bar z^l, \bar z^{l*}, \dot{\bar z}^l_\lambda, \dot{\bar z}^{l*}_{\lambda*})}{dt}. \tag{2.9}$$

Now let $X^i_\varepsilon(z^1, z^{1*}, \dot z^1_\lambda, \dot z^{1*}_{\lambda*})$ be a vector field of C_n then using (2.5), (2.3) and (2.4) we have

$$\overset{v}{d}X^i_\varepsilon = \left[\frac{\partial X^i_\varepsilon}{\partial z^k} v^k + \frac{\partial X^i_\varepsilon}{\partial z^{k*}} v^{k*} + \frac{\partial X^i_\varepsilon}{\partial \dot z^l_\lambda}\frac{\partial v^l}{\partial z^j}\dot z^j_\lambda + \frac{\partial X^i_\varepsilon}{\partial \dot z^{l*}_{\lambda*}}\frac{\partial v^{l*}}{\partial z^{j*}}\dot z^{j*}_{\lambda*}\right] dt. \tag{2.10}$$

Since the transformation law for the vector field $X^i_\varepsilon(z^1, z^{1*}, \dot z^1_\lambda, \dot z^{1*}_{\lambda*})$ in C_n is

$$X^i_\varepsilon(z^l, z^{l*}, \dot z^l_\lambda, \dot z^{l*}_{\lambda*}) = \frac{\partial z^i}{\partial \bar z^j} X^j_\varepsilon(\bar z^l, \bar z^{l*}, \dot{\bar z}^l_\lambda, \dot{\bar z}^{l*}_{\lambda*}), \tag{2.11}$$

therefore, from (2.7)a, (2.8) and (2.11) we have

$$\overset{m}{d}X^i_\varepsilon = X^j_\varepsilon \frac{\partial v^i}{\partial z^j} dt. \tag{2.12}$$

Hence by (2.9)

$$£_v X^i_\varepsilon = \frac{\partial X^i_\varepsilon}{\partial z^l} v^l + \frac{\partial X^i_\varepsilon}{\partial z^{l*}} v^{l*} + \frac{\partial X^i_\varepsilon}{\partial \dot z^l_\lambda}\frac{\partial v^l}{\partial z^j}\dot z^j_\lambda + \frac{\partial X^i_\varepsilon}{\partial \dot z^{l*}_{\lambda*}}\frac{\partial v^{l*}}{\partial z^{j*}}\dot z^{j*}_{\lambda*} - X^j_\varepsilon \frac{\partial v^i}{\partial z^j}. \tag{2.13}$$

If we consider the vector field $X^{i*}_{\varepsilon*}(z^1, z^{1*}, \dot z^1_\lambda, \dot z^{1*}_{\lambda*})$ then after using the transformation law

$$X^{i*}_{\varepsilon*}(z^l, z^{l*}, \dot z^l_\lambda, \dot z^{l*}_{\lambda*}) = \frac{\partial z^{i*}}{\partial \bar z^{j*}} X^{j*}_{\varepsilon*}(\bar z^l, \bar z^{l*}, \dot{\bar z}^l_\lambda, \dot{\bar z}^{l*}_{\lambda*}) \tag{2.14}$$

for $X^{i*}_{\varepsilon*}$ we obtain

$$£_v X^{i*}_{\varepsilon*} = \frac{\partial X^{i*}_{\varepsilon*}}{\partial z^l} v^l + \frac{\partial X^{i*}_{\varepsilon*}}{\partial z^{l*}} v^{l*} + \frac{\partial X^{i*}_{\varepsilon*}}{\partial \dot z^l_\lambda}\frac{\partial v^l}{\partial z^j}\dot z^j_\lambda + \frac{\partial X^{i*}_{\varepsilon*}}{\partial \dot z^{l*}_{\lambda*}}\frac{\partial v^{l*}}{\partial z^{j*}}\dot z^{j*}_{\lambda*} - X^{j*}_{\varepsilon*}\frac{\partial v^{i*}}{\partial z^{j*}}. \tag{2.15}$$

In a similar manner, the Lie derivative of covariant vector fields Y^ε_i and $Y^{\varepsilon*}_{i*}$ can be obtained. These are given by

$$£_v Y^\varepsilon_i = \frac{\partial Y^\varepsilon_i}{\partial z^l} v^l + \frac{\partial Y^\varepsilon_i}{\partial z^{l*}} v^{l*} + \frac{\partial Y^\varepsilon_i}{\partial \dot z^l_\lambda}\frac{\partial v^l}{\partial z^j}\dot z^j_\lambda + \frac{\partial Y^\varepsilon_i}{\partial \dot z^{l*}_{\lambda*}}\frac{\partial v^{l*}}{\partial z^{j*}}\dot z^{j*}_{\lambda*} + Y^\varepsilon_j \frac{\partial v^j}{\partial z^i}, \tag{2.16}$$

and

$$\text{(2.17)}\qquad \pounds_v Y^{\varepsilon^*}_{i^*} = \frac{\partial Y^{\varepsilon^*}_{i^*}}{\partial z^l} v^l + \frac{\partial Y^{\varepsilon^*}_{i^*}}{\partial z^{l^*}} v^{l^*} + \frac{\partial Y^{\varepsilon^*}_{i^*}}{\partial \dot z^l_\lambda} \frac{\partial v^l}{\partial z^j} \dot z^j_\lambda + + \frac{\partial Y^{\varepsilon^*}_{i^*}}{\partial \dot z^{l^*}_{\lambda^*}} \frac{\partial v^{l^*}}{\partial z^{j^*}} \dot z^{j^*}_{\lambda^*} + Y^{\varepsilon^*}_{i^*} \frac{\partial v^{j^*}}{\partial z^{i^*}}.$$

For a scalar $S(z^1, z^{1^*}, \dot z^1_\lambda, \dot z^{1^*}_{\lambda^*})$ we have

$$\text{(2.18)}\qquad \pounds_v S = \frac{\partial S}{\partial z^l} v^l + \frac{\partial S}{\partial z^{l^*}} v^{l^*} + \frac{\partial S}{\partial \dot z^l_\lambda} \frac{\partial v^l}{\partial z^j} \dot z^j_\lambda + \frac{\partial S}{\partial \dot z^{l^*}_{\lambda^*}} \frac{\partial v^{l^*}}{\partial z^{j^*}} \dot z^{j^*}_{\lambda^*}.$$

The theorem (2.1) given below is a direct consequence of equations (2.13), (2.15), (2.16), (2.17), (2.18) and the relation

$$S = X^i_\varepsilon Y^\varepsilon_i = X^{i^*}_{\varepsilon^*} Y^{\varepsilon^*}_{i^*}.$$

Theorem (2.1). *The Lie derivative defined by* (2.9) *satisfies the Leimbnitz rule i.e.*

$$\text{(2.19)}\qquad \pounds_v (X^i_\varepsilon Y^\varepsilon_i) = X^i_\varepsilon (\pounds_v Y^\varepsilon_i) + (\pounds_v X^i_\varepsilon) Y^\varepsilon_i,$$

$$\text{(2.20)}\qquad \pounds_v (X^{i^*}_{\varepsilon^*} Y^{\varepsilon^*}_{i^*}) = X^{i^*}_{\varepsilon^*} (\pounds_v Y^{\varepsilon^*}_{i^*}) + (\pounds_v X^{i^*}_{\varepsilon^*}) Y^{\varepsilon^*}_{i^*}.$$

The Lie derivatives of the partial derivative of X^i_ε with respect to $\dot z^1_\alpha$ and $\dot z^{1^*}_{\alpha^*}$ can be obtained from the first principle and we have

$$\text{(2.21)}\qquad \pounds_v \left(\frac{\partial X^i_\varepsilon}{\partial \dot z^l_\alpha}\right) = \frac{\partial^2 X^i_\varepsilon}{\partial z^k \partial \dot z^l_\alpha} v^k + \frac{\partial^2 X^i_\varepsilon}{\partial z^{k^*} \partial \dot z^l_\alpha} v^{k^*} + \frac{\partial^2 X^i_\varepsilon}{\partial \dot z^k_\beta \partial \dot z^l_\alpha} \frac{\partial v^k}{\partial z^j} \dot z^j_\beta + + \frac{\partial^2 X^i_\varepsilon}{\partial \dot z^{k^*}_{\beta^*} \partial \dot z^l_\alpha} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot z^{j^*}_{\beta^*} - \frac{\partial X^h_\varepsilon}{\partial \dot z^l_\alpha} \frac{\partial v^i}{\partial z^h} + \frac{\partial X^i_\varepsilon}{\partial \dot z^h_\alpha} \frac{\partial v^h}{\partial z^l} = = \frac{\partial}{\partial \dot z^l_\alpha} \left[\frac{\partial X^i_\varepsilon}{\partial z^k} v^k + \frac{\partial X^i_\varepsilon}{\partial z^{k^*}} v^{k^*} + \frac{\partial X^i_\varepsilon}{\partial \dot z^k_\beta} \frac{\partial v^k}{\partial z^j} \dot z^j_\beta + + \frac{\partial X^i_\varepsilon}{\partial \dot z^{k^*}_{\beta^*}} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot z^{j^*}_{\beta^*} - X^h_\varepsilon \frac{\partial v^i}{\partial z^h}\right] = \frac{\partial}{\partial \dot z^l_\alpha} (\pounds_v X^i_\varepsilon),$$

$$\text{(2.22)}\qquad \pounds_v \left(\frac{\partial X^i_\varepsilon}{\partial \dot z^{l^*}_{\alpha^*}}\right) = \frac{\partial^2 X^i_\varepsilon}{\partial z^k \partial \dot z^{l^*}_{\alpha^*}} v^k + \frac{\partial^2 X^i_\varepsilon}{\partial z^{k^*} \partial \dot z^{l^*}_{\alpha^*}} v^{k^*} + \frac{\partial^2 X^i_\varepsilon}{\partial \dot z^k_\beta \partial \dot z^{l^*}_{\alpha^*}} \frac{\partial v^k}{\partial z^j} z^j_\beta + + \frac{\partial^2 X^i_\varepsilon}{\partial \dot z^{k^*}_{\beta^*} \partial \dot z^{l^*}_{\alpha^*}} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot z^{j^*}_{\beta^*} - \frac{\partial X^h_\varepsilon}{\partial \dot z^{l^*}_{\alpha^*}} \frac{\partial v^i}{\partial z^h} + \frac{\partial X^i_\varepsilon}{\partial \dot z^{h^*}_{\alpha^*}} \frac{\partial v^{h^*}}{\partial z^{l^*}} = = \frac{\partial}{\partial \dot z^{l^*}_{\alpha^*}} \left[\frac{\partial X^i_\varepsilon}{\partial z^k} v^k + \frac{\partial X^i_\varepsilon}{\partial z^{k^*}} v^{k^*} + \frac{\partial X^i_\varepsilon}{\partial \dot z^k_\beta} \frac{\partial v^k}{\partial z^j} \dot z^j_\beta + + \frac{\partial X^i_\varepsilon}{\partial \dot z^{k^*}_{\beta^*}} \frac{\partial v^{k^*}}{\partial z^{j^*}} \dot z^{j^*}_{\beta^*} - X^h_\varepsilon \frac{\partial v^i}{\partial z^h}\right] = \frac{\partial}{\partial \dot z^{l^*}_{\alpha^*}} (\pounds_v X^i_\varepsilon).$$

Hence we have the following:

Theorem (2.2). *The operations* $\pounds_v$ *and* $\dfrac{\partial}{\partial \dot{z}^l_\alpha}$ *are commutative.*

Theorem (2.3). *The operations* $\pounds_v$ *and* $\dfrac{\partial}{\partial \dot{z}^{l^*}_{\alpha^*}}$ *are commutative.*

3. *Lie derivative of a vector field in terms of covariant partial derivatives.*

Since the transformation vector (v^i, v^{i^*}) depends only on z^1 and z^{1^*} we have from (1.19), (1.22), (1.23) and (1.24)

$$\text{(3.1)} \qquad \text{(a) } v^i_{|j} = \frac{\partial v^i}{\partial z^j} + \Gamma^i_{jl} v^l \quad \text{(b) } v^{i^*}_{|j^*} = \frac{\partial v^{i^*}}{\partial z^{j^*}} + \Gamma^{i^*}_{j^* l^*} v^{l^*},$$

$$\text{(3.2)} \qquad \text{(a) } v^i_{\|j} = \frac{\partial v^i}{\partial z^j} + \Gamma^i_{lj} v^l \quad \text{(b) } v^{i^*}_{\|j^*} = \frac{\partial v^{i^*}}{\partial z^{j^*}} + \Gamma^{i^*}_{l^* j^*} v^{l^*}.$$

Substituting (1.19), (1.20), (3.1)a, (3.1)b and (3.2)a in (2.13) we get (after some simplification)

$$\text{(3.3)} \qquad \pounds_v X^i_\varepsilon = X^i_{\varepsilon|k} v^k + X^i_{\varepsilon|k^*} v^{k^*} + \frac{\partial X^i_\varepsilon}{\partial \dot{z}^l_\alpha} v^l_{|j} \dot{z}^j_\alpha + \frac{\partial X^i_\varepsilon}{\partial \dot{z}^{l^*}_{\alpha^*}} v^{l^*}_{|j^*} \dot{z}^{j^*}_{\alpha^*} - X^j_\varepsilon v^i_{\|j}.$$

Similarly from (2.15), (1.21), (1.22), (3.1)a, (3.1)b and (3.2)b we have

$$\text{(3.4)} \qquad \pounds_v X^{i^*}_{\varepsilon^*} = X^{i^*}_{\varepsilon^*|k} v^k + X^{i^*}_{\varepsilon^*|k^*} v^{k^*} + \frac{\partial X^{i^*}_{\varepsilon^*}}{\partial \dot{z}^l_\alpha} v^l_{|j} \dot{z}^j_\alpha + \frac{\partial X^{i^*}_{\varepsilon^*}}{\partial \dot{z}^{l^*}_{\alpha^*}} v^{l^*}_{|j^*} \dot{z}^{j^*}_{\alpha^*} - X^{j^*}_{\varepsilon^*} v^{i^*}_{\|j^*}.$$

Generalizing this we may express the Lie derivative of an arbitrary tensor

$$T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*}$$

of C_n in the form

$$\begin{aligned}
\text{(3.5)} \qquad \pounds_v T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*} &= v^k T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^* | k} + \\
&+ v^{k^*} T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^* | k^*} + v^l_{|j} \dot{z}^j_\alpha \frac{\partial}{\partial \dot{z}^l_\alpha} T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*} + \\
&+ v^{l^*}_{|j^*} \dot{z}^{j^*}_{\alpha^*} \frac{\partial}{\partial \dot{z}^{l^*}_{\alpha^*}} T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*} - \\
&- \sum_v T^{i_1 \dots i_{v-1} k i_{v+1} \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*} v^{i_v}_{\|k} - \\
&- \sum_\mu T^{i_1 \dots i_r j_1^* \dots j_{\mu-1}^* k^* j_{\mu+1}^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_q^*} v^{j_\mu^*}_{\|k^*} + \\
&+ \sum_\lambda T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_{\lambda-1} k l_{\lambda+1} \dots l_p m_1^* \dots m_q^*} v^k_{\|l_\lambda} + \\
&+ \sum_\theta T^{i_1 \dots i_r j_1^* \dots j_s^* y_1 \dots y_p \delta_1^* \dots \delta_q^*}_{\alpha_1 \dots \alpha_r \beta_1^* \dots \beta_s^* l_1 \dots l_p m_1^* \dots m_{\theta-1}^* k^* m_{\theta+1}^* \dots m_q^*} v^{k^*}_{\|m_\theta^*}.
\end{aligned}$$

In particular the Lie derivative of the metric tensor is given by equations

$$\pounds_v g_{ij}^{\alpha\beta} = g_{ij|k}^{\alpha\beta} v^k + g_{ij|k^*}^{\alpha\beta} v^{k^*} + \frac{\partial g_{ij}^{\alpha\beta}}{\partial \dot z_\lambda^l} v_{|p}^l \dot z_\lambda^p + \tag{3.6}$$

$$+ \frac{\partial g_{ij}^{\alpha\beta}}{\partial \dot z_{\lambda^*}^{l^*}} v_{|p^*}^{l^*} \dot z_{\lambda^*}^{p^*} - g_{kj}^{\alpha\beta} v_{\|i}^k - g_{ik}^{\alpha\beta} v_{\|j}^k,$$

$$\pounds_v g_{ij^*}^{\alpha\beta^*} = g_{ij^*|k}^{\alpha\beta^*} v^k + g_{ij^*|k^*}^{\alpha\beta^*} v^{k^*} + \frac{\partial g_{ij^*}^{\alpha\beta^*}}{\partial \dot z_\lambda^l} v_{|p}^l \dot z_\lambda^p + \tag{3.7}$$

$$+ \frac{\partial g_{ij^*}^{\alpha\beta^*}}{\partial \dot z_{\lambda^*}^{l^*}} v_{|p^*}^{l^*} \dot z_{\lambda^*}^{p^*} - g_{kj^*}^{\alpha\beta^*} v_{\|i}^k - g_{ik^*}^{\alpha\beta^*} v_{\|j^*}^{k^*},$$

$$\pounds_v g_{i^*j^*}^{\alpha^*\beta^*} = g_{i^*j^*|k}^{\alpha^*\beta^*} v^k + g_{i^*j^*|k^*}^{\alpha^*\beta^*} v^{k^*} + \frac{\partial g_{i^*j^*}^{\alpha^*\beta^*}}{\partial \dot z_\lambda^l} v_{|p}^l \dot z_\lambda^p + \tag{3.8}$$

$$+ \frac{\partial g_{i^*j^*}^{\alpha^*\beta^*}}{\partial \dot z_{\lambda^*}^{l^*}} v_{|p^*}^{l^*} \dot z_{\lambda^*}^{p^*} - g_{k^*j^*}^{\alpha^*\beta^*} v_{\|i^*}^{k^*} - g_{i^*k^*}^{\alpha^*\beta^*} v_{\|j^*}^{k^*}.$$

4. *The Lie derivative of connection coefficient*

The Lie derivative of Γ_{kj}^i and $\Gamma_{k^*j^*}^{i^*}$ cannot be found directly from (3.5) as these are not components of a tensor. We shall, however, evaluate it from the first principle.

By (2.5) we have

$$\overset{v}{d}\Gamma_{kj}^i = \left[\frac{\partial \Gamma_{kj}^i}{\partial z^l} v^l + \frac{\partial \Gamma_{kj}^i}{\partial z^{l^*}} v^{l^*} + \frac{\partial \Gamma_{kj}^i}{\partial \dot z_\alpha^l}\frac{\partial v^l}{\partial z^p} \dot z_\alpha^p + \frac{\partial \Gamma_{kj}^i}{\partial \dot z_{\alpha^*}^{l^*}}\frac{\partial v^{l^*}}{\partial z^{p^*}} \dot z_{\alpha^*}^{p^*}\right] dt. \tag{4.1}$$

The law of transformation of Γ_{kj}^i is given by [10]

$$\Gamma_{kj}^i(\bar z^l, \bar z^{l^*}, \dot{\bar z}_\lambda^l, \dot{\bar z}_{\lambda^*}^{l^*}) = \frac{\partial \bar z^i}{\partial z^p}\left[\frac{\partial^2 z^p}{\partial \bar z^k \partial \bar z^j} + \Gamma_{ts}^p \frac{\partial z^t}{\partial \bar z^k}\frac{\partial z^s}{\partial \bar z^j}\right]. \tag{4.2}$$

Substituting (2.6)a and (2.7)a in (4.2) we get (after some simplification)

$$\overset{m}{d}\Gamma_{kj}^i = \Gamma_{kj}^i(\bar z^l, \bar z^{l^*}, \dot{\bar z}_\lambda^l, \dot{\bar z}_{\lambda^*}^{l^*}) - \Gamma_{kj}^i(z^l, z^{l^*}, \dot z_\lambda^l, \dot z_{\lambda^*}^{l^*}) = \tag{4.3}$$

$$= -\left[\frac{\partial^2 v^i}{\partial z^k \partial z^j} - \frac{\partial v^i}{\partial z^l}\Gamma_{kj}^l + \frac{\partial v^l}{\partial z^k}\Gamma_{lj}^i + \frac{\partial v^l}{\partial z^j}\Gamma_{kl}^i\right] dt.$$

Using the definition of Lie derivative and equations (4.1), (4.3) we have

$$\pounds_v \Gamma_{kj}^i = \left(\frac{\partial \Gamma_{kj}^i}{\partial z^l} - \frac{\partial \Gamma_{kj}^i}{\partial \dot z_\alpha^h}\Gamma_{pl}^h \dot z_\alpha^p\right) v^l + \left(\frac{\partial \Gamma_{kj}^i}{\partial z^{l^*}} - \frac{\partial \Gamma_{kj}^i}{\partial \dot z_{\alpha^*}^{h^*}}\Gamma_{p^*l^*}^{h^*} \dot z_{\alpha^*}^{p^*}\right) v^{l^*} + \tag{4.4}$$

$$+ \frac{\partial \Gamma_{kj}^i}{\partial \dot z_\alpha^h}\left(\frac{\partial v^h}{\partial z^p} + \Gamma_{pl}^h v^l\right)\dot z_\alpha^p + \frac{\partial \Gamma_{kj}^i}{\partial \dot z_{\alpha^*}^{h^*}}\left(\frac{\partial v^{h^*}}{\partial z^{p^*}} + \Gamma_{p^*l^*}^{h^*} v^{l^*}\right)\dot z_{\alpha^*}^{p^*} +$$

$$+ \frac{\partial^2 v^i}{\partial z^k \partial z^j} - \frac{\partial v^i}{\partial z^l}\Gamma_{kj}^l + \frac{\partial v^l}{\partial z^k}\Gamma_{lj}^i + \frac{\partial v^l}{\partial z^j}\Gamma_{kl}^i.$$

To express the Lie derivative of Γ^i_{kj} in terms of curvature tensor of C_n we consider the expansion of $v^i_{|k|j}$ noting that v^i is independent of $\dot{z}^1_\lambda$ and $\dot{z}^{1*}_{\lambda*}$. Therefore,

$$v^i_{|k|j} = v^l\left[\frac{\partial\Gamma^i_{kl}}{\partial z^j} - \frac{\partial\Gamma^i_{kl}}{\partial \dot{z}^p_\alpha}\Gamma^p_{mj}\dot{z}^m_\alpha + \Gamma^i_{jp}\Gamma^p_{kl}\right] + \frac{\partial^2 v^i}{\partial z^j \partial z^k} + \tag{4.5}$$

$$+\Gamma^i_{kl}\frac{\partial v^l}{\partial z^j} + \frac{\partial v^l}{\partial z^k}\Gamma^i_{jl} - \frac{\partial v^i}{\partial z^l}\Gamma^l_{jk} - \Gamma^p_{jk}\Gamma^i_{pl}v^l.$$

The curvature tensors R^i_{kjl} and R^i_{kjl*} for the connection coefficient Γ^i_{kj} are defined by

$$R^i_{kjl} = \left(\frac{\partial\Gamma^i_{kj}}{\partial z^l} - \frac{\partial\Gamma^i_{kj}}{\partial \dot{z}^h_\alpha}\Gamma^h_{pl}\dot{z}^p_\alpha\right) - \left(\frac{\partial\Gamma^i_{kl}}{\partial z^j} - \frac{\partial\Gamma^i_{kl}}{\partial \dot{z}^h_\alpha}\Gamma^h_{pj}\dot{z}^p_\alpha\right) + \tag{4.6}$$

$$+\Gamma^p_{kj}\Gamma^i_{pl} - \Gamma^p_{kl}\Gamma^i_{pj},$$

$$R^i_{kjl*} = \frac{\partial\Gamma^i_{kj}}{\partial z^{l*}} - \frac{\partial\Gamma^i_{kj}}{\partial \dot{z}^{h*}_{\alpha*}}\Gamma^{h*}_{p*l*}\dot{z}^{p*}_{\alpha*}. \tag{4.7}$$

A simple calculation based on (4.4), (4.5), (4.6) and (4.7) yields

$$\pounds_v\Gamma^i_{kj} = v^i_{|k|j} + R^i_{kjl}v^l + R^i_{kjl*}v^{l*} + \frac{\partial\Gamma^i_{kj}}{\partial \dot{z}^h_\alpha}v^h_{|p}\dot{z}^p_\alpha + \tag{4.8}$$

$$+\frac{\partial\Gamma^i_{kj}}{\partial \dot{z}^{h*}_{\alpha*}}v^{h*}_{|p*}\dot{z}^{p*}_{\alpha*} + T^i_{lj}v^l_{|k} + T^l_{jk}v^i_{|l},$$

where

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \tag{4.9}$$

is the torson tensor associated with the connection Γ^i_{jk}.

In a similar manner we can obtain the Lie derivative of the connection coefficients Γ^{i*}_{k*j*}. This is given by

$$\pounds_v\Gamma^{i*}_{k*j*} = v^{i*}_{|k*|j*} + R^{i*}_{k*j*l}v^l + R^{i*}_{k*j*l*}v^{l*} + \tag{4.10}$$

$$+\frac{\partial\Gamma^{i*}_{k*j*}}{\partial \dot{z}^h_\alpha}v^h_{|p}\dot{z}^p_\alpha + \frac{\partial\Gamma^{i*}_{k*j*}}{\partial \dot{z}^{h*}_{\alpha*}}v^{h*}_{|p*}\dot{z}^{p*}_{\alpha*} + T^{i*}_{l*j*}v^{l*}_{|k*} + T^{l*}_{j*k*}v^{i*}_{|l*}$$

where

$$R^{i*}_{k*j*l*} = \left[\frac{\partial\Gamma^{i*}_{k*j*}}{\partial z^{l*}} - \frac{\partial\Gamma^{i*}_{k*j*}}{\partial \dot{z}^{h*}_{\alpha*}}\Gamma^{h*}_{p*l*}\dot{z}^{p*}_{\alpha*}\right] - \left[\frac{\partial\Gamma^{i*}_{k*l*}}{\partial z^{j*}} - \frac{\partial\Gamma^{i*}_{k*l*}}{\partial \dot{z}^{h*}_{\alpha*}}\Gamma^{h*}_{p*j*}\dot{z}^{p*}_{\alpha*}\right] + \tag{4.11}$$

$$+\Gamma^{p*}_{k*j*}\Gamma^{i*}_{p*l*} - \Gamma^{p*}_{k*l*}\Gamma^{i*}_{p*j*},$$

$$R^{i*}_{k*j*l} = \frac{\partial\Gamma^{i*}_{k*j*}}{\partial z^l} - \frac{\partial\Gamma^{i*}_{k*j*}}{\partial \dot{z}^h_\alpha}\Gamma^h_{pl}\dot{z}^p_\alpha, \tag{4.12}$$

$$T^{i*}_{j*k*} = \Gamma^{i*}_{j*k*} - \Gamma^{i*}_{k*j*}. \tag{4.13}$$

5. *Areal motion.*

In this section we introduce the concept of an areal motion. When the fundamental metric function $L(z^1, z^{1^*}, \dot{z}^1_\lambda, \dot{z}^{1^*}_{\lambda^*})$ satisfies the relation

$$\pounds_v L = 0 \tag{5.1}$$

then the transformations (2.1)a and (2.1)b does not change the area

$$A = \iint_{(m)} L(z^l, z^{l^*}, \dot{z}^l_\lambda, \dot{z}^{l^*}_{\lambda^*})\, du^1 \wedge \ldots \wedge du^m \wedge du^{1^*} \wedge \ldots \wedge du^{m^*},$$

of an m-dimensional complex subspace ($2m$ dimensional real subspace). On account of this reason we give the following definition.

Definition (5.1). The transformation given by (2.1)a, (2.1)b is called an areal motion if $\pounds_v L = 0$.

Theorem (5.1). *In order that the space admits an areal motion it is necessary and sufficient that the Lie derivatives of the metric tensor* $g^{\beta\alpha^*}_{kj^*}$ *vanishes.*

PROOF. The necessary condition follows from theorems (2.2), (2.3) and the equations (5.1) and (1.12).

The sufficient part follows from theorem (2.1), the equation (1.16) and the facts $\pounds_v \dot{z}^h_\beta = 0$, $\pounds_v \dot{z}^{j^*}_{\alpha^*} = 0$.

Theorem (5.2). *If the space admits an areal motion then* $\pounds_v g^{\beta\alpha}_{hj} = 0$ *and* $\pounds_v g^{\beta^*\alpha^*}_{h^*j^*} = 0$.

The proof of the theorem follows from (1.11), (1.13), (5.1) and the theorems (2.2), (2.3).

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