Cyclically simple tournaments

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In [1], § 6, A. ÁDÁM introduced the concept of a cyclically simple directed graph. A directed graph G is called cyclically simple, if and only if the intersection of any two cycles in G is either empty, or a path. (A path of the length zero, i.e. formed by a single vertex, is also admitted.) A. Ádám proposed the problem to characterize cyclically simple graphs.

Here we shall solve this problem for finite tournaments. First we shall state some lemmas.

Lemma 1. *Every strongly connected finite tournament with $n \geqq 3$ vertices has a Hamiltonian cycle.*

PROOF. We shall use the induction according to the number n of vertices of a tournament. For $n=3$ a strongly connected tournament with n vertices is a cycle of the length 3 and thus the assertion holds. Suppose that the assertion is true for each n $(3 \leqq n \leqq k-1)$, where $k \geqq 4$. Let T be a strongly connected tournament with k vertices. Choose a vertex v of T. The tournament obtained from T by deleting v will be denoted by T'.

Case 1: T' is strongly connected. Then, according to the induction hypothesis, it contains a Hamiltonian cycle C'. The vertex v can be neither a source, nor a sink of T; otherwise T would not be strongly connected. Therefore the set of vertices of T' can be partitioned into two non-empty sets S^+, S^-, where S^+ is the set of all terminal vertices of edges outgoing from v in T and S^- is the set of all initial vertices of edges incoming into v in T. If we go around C', starting in a vertex of S^-, then let u_2 be the first vertex of S^+ into which we come and u_1 the initial vertex of the edge of C' incoming into u_2. (Obviously $u_1 \in S^-$.) If we delete the edge $\overrightarrow{u_1u_2}$ from C' and add the vertex v and the edges $\overrightarrow{u_1v}, \overrightarrow{vu_2}$, we obtain a Hamiltonian cycle of T.

Case 2: T' is not strongly connected. The set of all quasicomponents of a tournament can be linearly ordered by putting $Q_1 \prec Q_2$ if and only if there exists an edge going from a vertex of the quasicomponent Q_1 into a vertex of the quasicomponent Q_2 and $Q_1 \neq Q_2$. Let the quasicomponents of T' be $Q_1, \ldots, Q_r$, let $Q_i \prec Q_j$ for $1 \leqq i < j \leqq r$. Each Q_i for $i=1, \ldots, r$ is a strongly connected tournament with less than k vertices. Its number of vertices differs from 2, because a tournament with two vertices cannot be strongly connected. Thus either it consists of a single vertex, or (by the induction hypothesis) it contains a Hamiltonian cycle. If there exists such a cycle in Q_i $(i=1, \ldots, r)$, we denote it by C_i. Consider again the partition $\{S^+, S^-\}$. As T is strongly connected, there exists a vertex $x_1 \in S^+$ in Q_1 and a vertex $y_r \in S^-$ in Q_r.

If there exists C_1, then we denote by y_1 the initial vertex of the edge of C_1 whose terminal vertex is x_1; otherwise we put $y_1 = x_1$. If there exists C_r, then we denote by x_r the terminal vertex of the edge of C_r whose initial vertex is y_r; otherwise we put $x_r = y_r$. In each Q_i for $i = 2, \ldots, r-1$ we choose an arbitrary vertex x_i. If C_i exists, then we denote by y_i the initial vertex of the edge of C_i whose terminal vertex is x_i; otherwise we put $y_i = x_i$. Now consider a cycle C whose edges are $\overrightarrow{y_r v}, \overrightarrow{v x_1}, \overrightarrow{y_i x_{i+1}}$ for $i = 1, \ldots, r-1$ and further all edges of C_i except $\overrightarrow{y_i x_i}$ for each i for which C_i exists. The cycle C is a Hamiltonian cycle of T.

By a diagonal edge of a cycle C in a directed graph we shall mean an edge joining two vertices of C and not belonging to C.

Lemma 2. *Let G be a cyclically simple directed graph, let C be a cycle in G with the vertices $u_1, \ldots, u_k$ and edges $\overrightarrow{u_i u_{i+1}}$ for $i = 1, \ldots, k-1$ and $\overrightarrow{u_k u_1}$. If in G there exists a diagonal edge $\overrightarrow{u_1 u_p}$ of C for some p, $3 \leqq p \leqq k-1$, then the edge $\overrightarrow{u_q u_1}$ for $p+1 \leqq q \leqq k-1$ does not exist in G.*

Proof. Suppose that there exist both edges $\overrightarrow{u_1 u_p}, \overrightarrow{u_q u_1}$. Let C' be the cycle whose edges are $\overrightarrow{u_q u_1}, \overrightarrow{u_1 u_p}$ and all edges of the path P in C from u_p into u_q (as $p < q$, this path does not contain u_1). The intersection of C and C' consists of the path P and of the vertex u_1, which is a contradiction.

The mentioned "forbidden" situation is in Fig. 1.

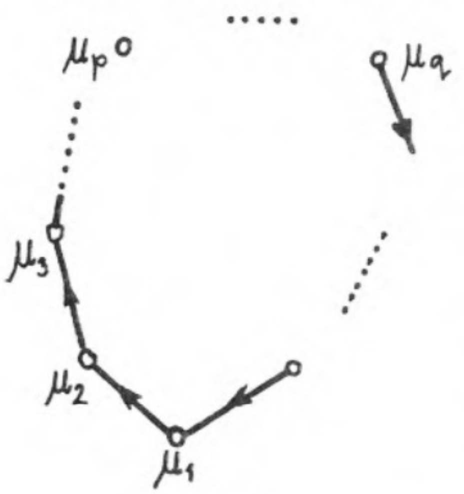

Fig. 1.

Lemma 3. *A directed graph G is cyclically simple if and only if all of its quasicomponents are cyclically simple.*

Proof. This assertion follows immediately from the fact that each cycle of G lies in some quasicomponent of G and cycles lying in distinct quasicomponents of G have an empty intersection.

Now we shall prove a theorem.

Theorem. *Let T be a finite tournament. Then the following two assertions are equivalent:*

(i) *T is cyclically simple.*

(ii) *Let Q be an arbitrary quasicomponent of T with at least three vertices. Then there exists a notation of (all) vertices of Q by $u_1, u_2, \ldots, u_k$ fulfilling the subsequent statements:*

(a) *the edges $\overrightarrow{u_i u_{i+1}}$ exist for $i = 1, \ldots, k-1$,*

(b) *the edge $\overrightarrow{u_k u_1}$ exists,*

(c) *whenever $p - q \not\equiv \pm 1 \pmod{k}$ and $p > q$, then the edge $\overrightarrow{u_p u_q}$ exists in Q.*

Remark. The condition stated in (ii) for Q may be formulated in the subsequent manner (equivalently): there exists a directed path P containing all the vertices of Q such that if we reverse the orientation of the edges of P, then we obtain a transitively directed tournament.

PROOF. (i)$\Rightarrow$(ii). Let T be a cyclically simple tournament. Each quasicomponent Q of T is a strongly connected tournament, therefore either it consists of a single vertex, or contains a Hamiltonian cycle (by Lemma 1). Let C be such a Hamiltonian cycle. Let Q' be the graph obtained from Q by deleting all edges of C. If Q' contains a cycle C', then the intersection of C and C' consists of all vertices of C' and no edges, which is a contradiction with the assumption that T is cyclically simple. Therefore Q' must be acyclic and, being finite, it contains a sink; denote it by u_1. Other vertices of Q will be denoted by $u_2, \dots, u_k$ so that the edges of C might be $\overrightarrow{u_i u_{i+1}}$ for $i=1, \dots, k-1$ and $\overrightarrow{u_k u_1}$. Then the edge $\overrightarrow{u_p u_1}$ exists for any $p=3, \dots, k-1$; thus for $q=1$ the condition is fulfilled. Now let p, q be integers such that $2 \leqq q < q+2 \leqq p \leqq k-1$. There cannot exist the edge $\overrightarrow{u_q u_p}$, because the existence of both the edges $\overrightarrow{u_q u_p}, \overrightarrow{u_p u_1}$ would contradict to Lemma 2. As T is a tournament, there exists the edge $\overrightarrow{u_p u_q}$ and the condition is fulfilled for all considered pairs $\{p, q\}$, where $p \neq k$. Now we shall consider diagonal edges of C which are incident with u_k. Suppose that there exist integers r, s such that $2 \leqq r \leqq k-2$, $2 \leqq s \leqq k-2$ and $\overrightarrow{u_r u_k}, \overrightarrow{u_k u_s}$ are edges of Q. By Lemma 2 we have $r < s$. Then there exists a cycle C'' having edges $\overrightarrow{u_r u_k}, \overrightarrow{u_k u_s}, \overrightarrow{u_s u_{s+1}}, \overrightarrow{u_{s+1} u_r}$. The intersection of C and C'' consists of the edge $\overrightarrow{u_s u_{s+1}}$ with its initial and terminal vertices and of the vertices u_k, u_r; therefore T is not cyclically simple, which is a contradiction. Hence either u_k is the initial vertex of all diagonal edges of C which are incident with it, or it is the terminal vertex of all of them. In the first case the assertion is proved. In the second case we change the notation so that u_i will be replaced by u_{i+1} for $i=1, \dots, k-1$ and u_k will be replaced by u_1; then the assertion will be true.

(ii)$\Rightarrow$(i). Let T fulfil (ii). By Lemma 3 it suffices to prove that each quasicomponent Q of T is cyclically simple. If Q consists of a single vertex, this is evident. Consider the other case. Let C be the cycle of Q whose vertices are $u_1, u_2, \dots, u_k$ and edges $\overrightarrow{u_i u_{i+1}}$ for $i=1, \dots, k-1$ and $\overrightarrow{u_k u_1}$. Let C_0 be a cycle in C distinct from C. Let Q' be defined as in the first part of the proof. As Q' is acyclic (which can be easily proved), one connected component of the intersection of C and C_0 must be a directed path P of a non-zero length. Let its initial vertex be u_a and its terminal vertex u_b.

Case 1: $a < b$. The edge of C_0 incoming into u_a does not belong to C, because otherwise P would not be a connected component of the intersection of C and C_0. Thus its initial vertex is u_c for some $c > a$. Analogously the terminal vertex of the edge of C_0 outgoing from u_b is u_d for some $d < b$. If $c=b$, then also $d=a$ and the intersection of C and C_0 is P. If $c \neq b$, then evidently $c > b$, because otherwise u_c would belong to P. (If $b=k$, then c evidently cannot exist.) Then also $d < a$. The cycle C_0 must contain a directed path from u_d into u_c disjoint with P. We are going to show that such a path does not exist. By deleting the vertices of P from Q we obtain a graph having two quasicomponents; one having the vertices $u_1, \dots, u_{a-1}$, the other with the vertices $u_{b+1}, \dots, u_k$.

Each edge joining the vertices of distinct quasicomponents goes from the second into the first. Therefore this case is impossible.

Case 2: $b<a$. Then P contains all vertices u_p for $a \leqq p \leqq k$. Let again u_c be the initial vertex of the edge of C_0 whose terminal vertex is u_a. Then again $c>a$ and thus u_c belongs to P, which is impossible. The assertion is proved.

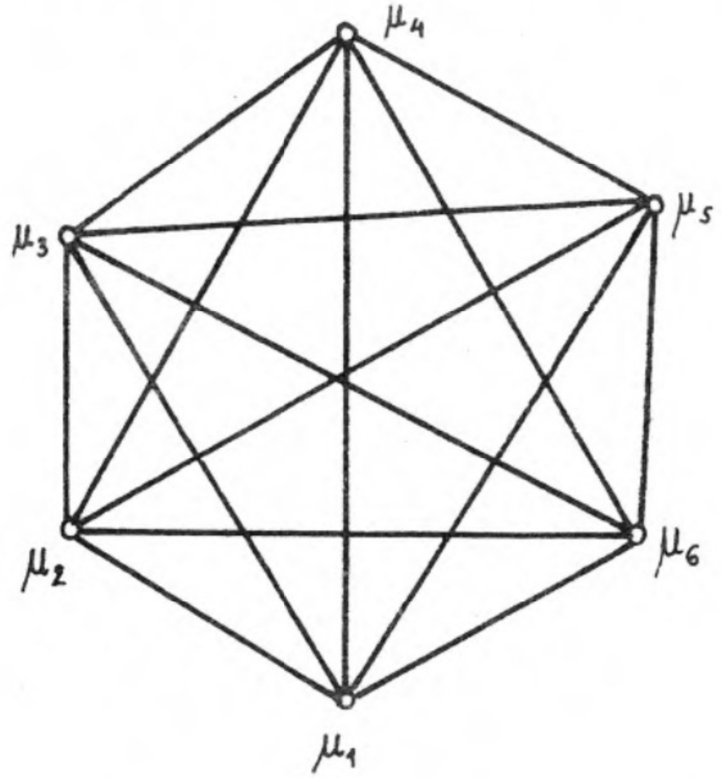

Fig. 2.

Fig. 2 shows a cyclically simple strongly connected tournament with six vertices.

Reference

[1] A. ÁDÁM, On some open problems of applied automaton theory and graph theory (suggested by the mathematical modelling of certain neuronal networks), *Acta Cybernetica (Szeged)* 3 (1977), 187—214.

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