On projective invariants based on non-linear connections in a Finsler space

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Non-linear connections and their properties in a Finsler space have been studied by various authors namely VAGNER [9], KAWAGUCHI [2, 3, 4], BARTHEL [1], RUND [7, 8], MISRA and MISHRA [5] and RASTOGI [6] etc. The purpose of the present paper is to study the effect of projective transformation on non-linear connections in a Finsler space. We have also obtained certain entities which are projectively invariant.

1. *Introduction.* Let $X^i(x^k)$ and $Y_j(x^k)$ be two differentiable vector fields in a Finsler space F_n, with metric tensor $g_{ij}(x, X)$ and non-linear connections $\overset{1}{\Gamma}{}^i_k(x, X)$ and $\overset{2}{\Gamma}_{jk}(x, Y)$, positively homogeneous of first degree in X and Y respectively, then we have RUND [7]:

$$\overset{1}{\Gamma}{}^i_{jk}(x, X) = \Delta_j \overset{1}{\Gamma}{}^i_k,\ X^j \overset{1}{\Gamma}{}^i_{jk} = \overset{1}{\Gamma}{}^i_k \tag{1.1}$$

and

$$\overset{2}{\Gamma}{}^i_{jk}(x, Y) = \Delta^i \overset{2}{\Gamma}_{jk},\ Y_i \overset{2}{\Gamma}{}^i_{jk} = \overset{2}{\Gamma}_{jk}, \tag{1.2}$$

where $\Delta_j = \partial/\partial X^j$ and $\Delta^i = \partial/\partial Y_i$.

Let us suppose that if X^i undergoes a parallel displacement then so does $Y_i = g_{ij} X^j$, such that the length of a vector remains unchanged under parallel displacement, then we have [7]:

$$2G^i = \overset{1}{\Gamma}{}^i_k X^k + g^{ih} Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h). \tag{1.3}$$

Assuming geodesics to be autoparallel curves of F_n we get

$$2G^i = \overset{1}{\Gamma}{}^i_k X^k, \tag{1.4}$$

which implies

$$\overset{1}{\Gamma}{}^i_j = 2G^i_j - \overset{1}{\Gamma}{}^i_{jh} X^h \tag{1.5}$$

and

$$\overset{1}{\Gamma}{}^i_{hj} = G^i_{hj} + \frac{1}{2} \{\overset{1}{S}{}^i_{hj} - X^k \Delta_j \overset{1}{S}{}^i_{hk}\}, \tag{1.6}$$

where

$$\overset{1}{S}{}^i_{hk} = 2\overset{1}{\Gamma}{}^i_{[hk]} = \overset{1}{\Gamma}{}^i_{hk} - \overset{1}{\Gamma}{}^i_{kh}.$$

Also the two non-linear connections are related by

$$\overset{2}{\Gamma}{}^h_{ik} = \overset{1}{\Gamma}{}^h_{ik} + Y_j \Delta^h \overset{1}{\Gamma}{}^j_{ik}. \tag{1.7}$$

The covariant derivative of a tensor field $T^i{}_j(x, X)$ is given by [5]:

$$T^i_{j,k}(x, X) = \partial_k T^i_j + (\Delta_m T^i_j)\partial_k X^m + T^m_j \overset{1}{\Gamma}{}^i_{mk} - T^i_m \overset{2}{\Gamma}{}^m_{jk}. \tag{1.8}$$

2. *Some curvature properties.* Since we know that [5]:

$$\overset{1}{R}{}^i_{kh} = 2\{\partial_{[h} \overset{1}{\Gamma}{}^i_{k]} + \overset{1}{\Gamma}{}^i_{m[h} \overset{1}{\Gamma}{}^m_{k]}\}, \tag{2.1}$$

therefore substituting from (1.5) and (1.6) in (2.1) we get on simplification

$$\overset{1}{R}{}^i_{kh} = 2\{H^i_{kh} + \overset{1}{\Gamma}{}^m_{[k} \overset{1}{\Gamma}{}^i_{h]m} - G^i_{[h|m|}(\overset{1}{\Gamma}{}^m_{k]l} X^l + \overset{1}{\Gamma}{}^m_{k]}) - \partial_{[h}(\overset{1}{\Gamma}{}^i_{k]l} X^l)\}, \tag{2.2}$$

where [8]:

$$H^i_{jk} = 2\{\partial_{[k} \Delta_{j]} G^i + (\Delta_{[j} G^r) G^i_{k]r}\}. \tag{2.3}$$

Multiplying (2.2) by $X^k X^h$ and using $H^i_{jk} X^j X^k = 0$, we get

$$\overset{1}{R}{}^i_{jk} X^j X^k = 0. \tag{2.4}$$

Now applying [5]:

$$\overset{1}{R}{}^i_{jkh} = \Delta_j \overset{1}{R}{}^i_{kh} + 2(\Delta_j \overset{1}{\Gamma}{}^i_{m[k}) X^m_{,h]}, \tag{2.5}$$

in equation (2.2) and using [8]:

$$H^i_{hjk} = 2\{\partial_{[k} G^i_{|h|j]} + G^i_{rh[k} \Delta_{j]} G^r + G^r_{h[j} G^i_{|r|k]}\}, \tag{2.6}$$

we get

$$\begin{aligned}\overset{1}{R}{}^i_{jkh} = 2\{&H^i_{jkh} + \overset{1}{\Gamma}{}^m_{[k} \Delta_j \overset{1}{\Gamma}{}^i_{h]m} + \overset{1}{\Gamma}{}^m_{j[k} \overset{1}{\Gamma}{}^i_{h]m} + \\ &+ (\Delta_j \overset{1}{\Gamma}{}^i_{m[k}) X^m_{,h]} - \Delta_j \partial_{[h}(\overset{1}{\Gamma}{}^i_{k]l} X^l) - G^i_{[h|m|}(2\overset{1}{\Gamma}{}^m_{(k]j)} + X^l \Delta_j \overset{1}{\Gamma}{}^m_{k]l})\}.\end{aligned} \tag{2.7}$$

Since we know that [8]:

$$H^i_{jkh} + H^i_{khj} + H^i_{hjk} = 0 \tag{2.8}$$

and Rastogi [6]:

$$(\overset{1}{R}{}^i_{jkh} - \overset{1}{S}{}^i_{jk,h} - \overset{1}{S}{}^i_{jl} \overset{2}{S}{}^l_{kh}) + \mathrm{cycl}\,(j, k, h) = 0, \tag{2.9}$$

where $\overset{2}{S}{}^l_{kh} = 2\overset{2}{\Gamma}{}^l_{[kh]}$, therefore from equation (2.7) we obtain on simplification

$$\begin{aligned}&\{\overset{1}{\Gamma}{}^m_{[k} \Delta_j \overset{1}{\Gamma}{}^i_{h]m} + \overset{1}{\Gamma}{}^m_{j[k} \overset{1}{\Gamma}{}^i_{h]m} + (\Delta_j \overset{1}{\Gamma}{}^i_{m[k}) X^m_{,h]} - \\ &- \Delta_j \partial_{[h}(\overset{1}{\Gamma}{}^i_{k]l} X^l) - G^i_{[h|m|}(2\overset{1}{\Gamma}{}^m_{(k]j)} + X^l \Delta_j \overset{1}{\Gamma}{}^m_{k]l}) - \overset{1}{\Gamma}{}^i_{[jk],h} - \overset{1}{\Gamma}{}^i_{[jl]} \overset{2}{S}{}^l_{kh}\} + \mathrm{cycl}\,(j, k, h) = 0.\end{aligned} \tag{2.10}$$

Since we know that [5]:

$$\overset{2}{R}{}_{jkh} = 2\{\partial_{[h} \overset{2}{\Gamma}{}_{|j|k]} + \overset{2}{\Gamma}{}^m_{j[k} \overset{2}{\Gamma}{}_{|m|h]}\}, \tag{2.11}$$

therefore by using equation (1.7) in (2.11) we obtain

$$\overset{2}{R}_{jkh} = Y_i \overset{1}{R}{}^i_{jkh} + 2\{(\partial_{[h} Y_i) \overset{1}{\Gamma}{}^i_{|j|k]} - (\Delta_m \overset{1}{\Gamma}{}^i_{j[k}) Y_i (\partial_{h]} X^m - \overset{2}{\Gamma}{}_{|r|h]} g^{rm})\}. \tag{2.12}$$

Differentiating (2.12) with respect to Y_1 and using [5]:

$$\overset{2}{R}{}^i_{jkh} = \Delta^i \overset{2}{R}_{jkh} + 2(\Delta^i \overset{2}{\Gamma}{}^m_{j[k}) Y_{|m|.h]}, \tag{2.13}$$

we get on simplification

$$\begin{aligned} \overset{2}{R}{}^l_{jkh} = {} & \overset{1}{R}{}^l_{jhk} + Y_i \Delta^l \overset{1}{R}{}^i_{jkh} + 2[(\partial_{[h} Y_i)(\Delta^l \overset{1}{\Gamma}{}^i_{|j|k]}) + \\ & + \overset{1}{\Gamma}{}^i_{j[k} \Delta^i (\partial_{h]} Y_i) - (\Delta^l \Delta_m \overset{1}{\Gamma}{}^i_{j[k})(Y_i \partial_{h]} X^m - Y_i \overset{2}{\Gamma}{}_{|p|h]} g^{pm}) - \\ & - \Delta_m \overset{1}{\Gamma}{}^l_{j[k} (\partial_{h]} X^m - \overset{2}{\Gamma}{}_{|p|h]} g^{pm}) - \Delta_m \overset{1}{\Gamma}{}^i_{j[k} Y_i \{\Delta^l \partial_{h]} X^m - \Delta^l (\overset{2}{\Gamma}{}_{|p|h]} g^{pm})\}]. \end{aligned} \tag{2.14}$$

Further by using (2.9) and [6]:

$$(\overset{2}{R}{}^i_{jkh} - \overset{2}{S}{}^i_{jk.h} - \overset{2}{S}{}^i_{jl} \overset{2}{S}{}^l_{kh}) + \text{cycl}\,(j, k, h) = 0, \tag{2.15}$$

one can establish a relationship analogous to (2.10).

3. *Projective transformation.* Let $P(x, X)$ be an arbitrary scalar function positively homogeneous of the first degree in X^i, then the projective change of G^i is given by [8]:

$$\bar{G}^i = G^i - P(x, X) X^i. \tag{3.1}$$

Transforming equation (1.4) projectively and using (3.1) we obtain

$$\overset{1}{\bar{\Gamma}}{}^i_k = \overset{1}{\Gamma}{}^i_k - 2P\delta^i_k. \tag{3.2}$$

Differentiating (3.2) with respect to X^j we get

$$\overset{1}{\bar{\Gamma}}{}^i_{jk} = \overset{1}{\Gamma}{}^i_{jk} - 2\delta^i_k \Delta_j P. \tag{3.3}$$

Further transforming equation (1.7) projectively and making use of (3.3) we obtain

$$\overset{2}{\bar{\Gamma}}{}^h_{ik} = \overset{2}{\Gamma}{}^h_{ik} - 2\delta^h_k \Delta_i P - 2Y_k (\Delta^h \Delta_i P). \tag{3.4}$$

Now defining $\overset{1}{\bar{R}}{}^i_{kh}$ corresponding to $\overset{1}{\bar{\Gamma}}{}^i_{jk}$ in analogy to (2.1) and making use of (3.2) and (3.3) we obtain after some simplification

$$\overset{1}{\bar{R}}{}^i_{kh} = \overset{1}{R}{}^i_{kh} + 2\delta^i_h \Phi_k - 2\delta^i_k \Phi_h - 2P\overset{1}{S}{}^i_{kh}, \tag{3.5}$$

where

$$\Phi_k = \partial_k P + 2P\Delta_k P - \overset{1}{\Gamma}{}^m_k \Delta_m P. \tag{3.6}$$

From equation (3.5) we establish:

Theorem (3.1). *A necessary and sufficient condition for the tensor* $\overset{1}{R}{}^i_{kh}$ *to be invariant is given by* $\overset{1}{\Gamma}{}^i_{kh} = \frac{1}{P}\delta^i_h \Phi_k$.

Multiplying equation (3.5) by $X^k X^h$ and using (2.4) one easily obtains

$$\overset{1}{\bar{R}}{}^i_{kh} X^k X^h = 0. \tag{3.7}$$

Putting $g^{kh}\overset{1}{R}{}^i_{kh} = \overset{1}{R}{}^i$ and multiplying equation (3.5) by g^{kh}, one easily observes $\overset{1}{\bar{R}}{}^i = \overset{1}{R}{}^i$.

Using equation (2.5) and defining $\overset{1}{\bar{R}}{}^i_{jkh}$ corresponding to $\overset{1}{\bar{R}}{}^i_{kh}$ we get by virtue of (3.3) and (3.5), the following relation:

$$\overset{1}{\bar{R}}{}^i_{jkh} = \overset{1}{R}{}^i_{jkh} + 2\{2\Phi_{j[k}\delta^i_{h]} - \Delta_j P \overset{1}{S}{}^i_{kh} + \tag{3.8}$$

$$+ (\Delta_j \overset{1}{\Gamma}{}^i_{m[k}) X^m_{,h]} - 2\delta^i_{[k} X^m_{,h]}(\Delta_j \Delta_m P) + 4P\delta^i_{[k}\Delta_j \Delta_{h]} P\},$$

where $\Phi_{jk} = \Delta_j \Phi_k$.

Now using equation (2.9) and observing that

$$\overset{1}{\bar{S}}{}^i_{jk} = \overset{1}{S}{}^i_{jk} - 4\delta^i_{[k}\Delta_{j]} P \tag{3.9}$$

and

$$\overset{2}{\bar{S}}{}^i_{jk} = \overset{2}{S}{}^i_{jk} - 4\{\delta^i_{[k}\Delta_{j]} P + Y_{[k}(\Delta^i \Delta_{j]} P)\}, \tag{3.10}$$

we get on simplification

$$[\delta^i_{[k}\{2P\Delta_j \Delta_{h]} P - X^m_{,h]}\Delta_j \Delta_m P - \Phi_{|j|h]} - (\Delta_{j]} P)_{,h} + 2\Delta_{h]} P\} - \tag{3.11}$$

$$- \overset{1}{S}{}^i_{j[k}\Delta_{h]} P - \overset{1}{S}{}^i_{jl} Y_{[k}\Delta^l \Delta_{h]} P - 2\Delta_j P Y_{[h}\Delta^l \Delta_{k]} P + 2\delta^i_j \Delta_l P Y_{[h}\Delta^l \Delta_{k]} P -$$

$$-2\{(\Delta_j P)\overset{1}{S}{}^i_{kh} - (\Delta_j \overset{1}{\Gamma}{}^i_{m[k}) X^m_{,h]} - \overset{2}{S}{}^i_{kh}\Delta_j P + \delta^i_j \Delta_l P \overset{2}{S}{}^l_{kh}\}] + \text{cycl}\,(j, k, h) = 0.$$

Defining a tensor $\overset{2}{\bar{R}}_{jkh}(x, Y)$ in anlogy to (2.11) and using equations (3.4) and

$$\overset{2}{\bar{\Gamma}}_{ik} = \overset{2}{\Gamma}_{ik} - 2Y_k(\Delta_i P), \tag{3.12}$$

we obtain

$$\overset{2}{\bar{R}}_{jkh}(x, Y) = \overset{2}{R}_{jkh} + 2\psi_{j[k} Y_{h]} - 4(\Delta_j P) Y_{[h,k]}, \tag{3.13}$$

where

$$\psi_{jk} = 2\{2\partial_k \Delta_j P + 2(\Delta_k P)(\Delta_j P) + \overset{2}{\Gamma}_{mk}\Delta^m \Delta_j P - \overset{2}{\Gamma}{}^m_{jk}\Delta_m P\}. \tag{3.14}$$

From equation (3.13) we easily establish:

Theorem (3.2). *A necessary and sufficient condition for the tensor* $\overset{2}{R}_{jkh}$ *to be invariant is given by* $Y_h \psi_{jk} = 2(\Delta_j P) Y_{h,k}$.

Further using equation (2.13) and defining $\overset{2}{\bar{R}}{}^{i}_{jkh}$ corresponding to $\overset{2}{\bar{R}}_{jkh}$ we obtain

$$(3.15)\qquad \overset{2}{\bar{R}}{}^{i}_{jkh} = \overset{2}{R}{}^{i}_{jkh} + 2\Delta^i(\psi_{j[k}Y_{h]}) - 4[\Delta^i(\Delta_j P Y_{[h,k]}) -$$
$$-(\Delta^i \overset{2}{\Gamma}{}^{m}_{j[k})Y_m \partial_{h]}P + \{\delta^m_{[k}\Delta^i\Delta_j P + \Delta^i(Y_{[k}\Delta^m\Delta_j P)\}Y_{|m|,h]}].$$

Remarks. i) A relation similar to (3.11) can also be established from equations (2.15) and (3.15).

ii) Identities analogous to Bianchi can also be established for the projectively transformed curvature tensors as in [6].

4. *Some projective invariants.* By contracting equations (3.2), (3.3) and (3.4) one can easily obtain the following equations

$$(4.1)\qquad \overset{1}{\bar{\Gamma}}{}^{i}_{i} = \overset{1}{\Gamma}{}^{i}_{i} - 2Pn,$$

$$(4.2)\qquad \overset{1}{\bar{\Gamma}}{}^{i}_{ik} = \overset{1}{\Gamma}{}^{i}_{ik} - 2\Delta_k P,$$

$$(4.3)\qquad \overset{1}{\bar{\Gamma}}{}^{i}_{ki} = \overset{1}{\Gamma}{}^{i}_{ki} - 2n\Delta_k P,$$

$$(4.4)\qquad \overset{2}{\bar{\Gamma}}{}^{h}_{ih} = \overset{2}{\Gamma}{}^{h}_{ih} - 2n\Delta_i P$$

and

$$(4.5)\qquad \overset{2}{\bar{\Gamma}}{}^{h}_{hk} = \overset{2}{\Gamma}{}^{h}_{hk} - 2\Delta_k P - 2Y_k(\Delta^h\Delta_h P).$$

By eliminating P from equations (3.2) and (4.1) we can easily define

$$(4.6)\qquad {}'\overset{1}{\Gamma}{}^{i}_{k} \overset{\text{def}}{=\!=} \overset{1}{\Gamma}{}^{i}_{k} - \frac{1}{n}\delta^i_k \overset{1}{\Gamma}{}^{l}_{l},$$

which is a projectively invariant entity such that ${}'\overset{1}{\Gamma}{}^{i}_{i}=0$.

Further by similar eliminations of P from (3.3), (3.4) and (4.2), (4.3), (4.4) and (4.5) we define the following projectively invariant entities:

$$(4.7)\qquad \overset{1}{\Gamma}{}^{*i}_{jk} \overset{\text{def}}{=\!=} \overset{1}{\Gamma}{}^{i}_{jk} - \delta^i_k \overset{1}{\Gamma}{}^{l}_{lj},$$

$$(4.8)\qquad {}'\overset{1}{\Gamma}{}^{i}_{jk} \overset{\text{def}}{=\!=} \overset{1}{\Gamma}{}^{i}_{jk} - \frac{1}{n}\delta^i_k \overset{1}{\Gamma}{}^{l}_{jl},$$

$$(4.9)\qquad \overset{1}{\tilde{\Gamma}}{}^{i}_{jk} \overset{\text{def}}{=\!=} \overset{1}{\Gamma}{}^{i}_{jk} - \frac{1}{n}\delta^i_k \overset{2}{\Gamma}{}^{l}_{jl},$$

$$(4.10)\qquad \overset{2}{\Gamma}{}^{*i}_{jk} \overset{\text{def}}{=\!=} \overset{2}{\Gamma}{}^{i}_{jk} - \delta^i_k \overset{1}{\Gamma}{}^{l}_{lj} - Y_k\Delta^i \overset{1}{\Gamma}{}^{l}_{lj},$$

$$(4.11)\qquad {}'\overset{2}{\Gamma}{}^{i}_{jk} \overset{\text{def}}{=\!=} \overset{2}{\Gamma}{}^{i}_{jk} - \frac{1}{n}\{\delta^i_k \overset{1}{\Gamma}{}^{l}_{jl} + Y_k\Delta^i \overset{1}{\Gamma}{}^{l}_{jl}\}$$

and

$$\overset{2}{\bar{\Gamma}}{}^i_{jk} \overset{\text{def}}{=} \overset{2}{\Gamma}{}^i_{jk} - \frac{1}{n}\{\delta^l_k \overset{2}{\Gamma}{}^l_{jl} + Y_k \Delta^i \overset{2}{\Gamma}{}^l_{jl}\}. \tag{4.12}$$

From the definitions of these invariants it is observed that the first three invariants are related with the last three in the same manner as $\overset{1}{\Gamma}{}^i_{jk}$ is related with $\overset{2}{\Gamma}{}^i_{jk}$. Also one can observe that

$$\overset{1}{\bar{\Gamma}}{}^i_{jk} = {}'\overset{1}{\Gamma}{}^i_{jk} - \frac{1}{n}\delta^i_k Y_r \Delta^l \overset{1}{\Gamma}{}^r_{jl} \tag{4.13}$$

and

$$\overset{2}{\bar{\Gamma}}{}^h_{ik} = {}'\overset{2}{\Gamma}{}^h_{ik} - \frac{1}{n}\{\delta^h_k Y_r \Delta^m \overset{1}{\Gamma}{}^r_{im} + Y_k \Delta^h (Y_r \Delta^m \overset{1}{\Gamma}{}^r_{im})\}. \tag{4.14}$$

Thus we have:

Theorem (4.1). *A necessary and sufficient condition for the invariants* $\overset{1}{\bar{\Gamma}}{}^i_{jk}$ *and* ${}'\overset{1}{\Gamma}{}^i_{jk}$ *to be identical is given by*

$$Y_r \Delta^l \overset{1}{\Gamma}{}^r_{im} = 0.$$

Remark. Some other invariants which are trivial can be obtained by eliminating P from (4.2), (4.3), (4.4) and (4.5).

Corresponding to these projective invariants defined by (4.7) to (4.12), it is possible to define three types of projective covariant derivatives. The covariant derivative of a tensor $T^i_j(x, X)$ for the projective connection $\overset{1}{\Gamma}{}^{*i}_{jk}$ is given by

$$T^i_{j|k} \overset{\text{def}}{=} \partial_k T^i_j + (\Delta_m T^i_j)\partial_k X^m + T^m_j \overset{1}{\Gamma}{}^{*i}_{mk} - T^i_m \overset{2}{\Gamma}{}^{*m}_{jk}. \tag{4.15}$$

Remark. Other two can be defined by replacing $\overset{1}{\Gamma}{}^{*i}_{jk}$ and $\overset{2}{\Gamma}{}^{*i}_{jk}$ by ${}'\overset{1}{\Gamma}{}^i_{jk}$, ${}'\overset{2}{\Gamma}{}^i_{jk}$ and $\overset{1}{\bar{\Gamma}}{}^i_{jk}$, $\overset{2}{\bar{\Gamma}}{}^i_{jk}$ and will be denoted by $T^i_{j;k}$ and $T^i_{j\|k}$ respectively.

From equations (1.8) and (4.15) we obtain

$$T^i_{j|k} = T^i_{j,k} + T^i_k \overset{1}{\Gamma}{}^l_{lj} + Y_k T^i_m \Delta^m \overset{1}{\Gamma}{}^l_{lj} - \delta^i_k T^m_j \overset{1}{\Gamma}{}^l_{lm}. \tag{4.16}$$

Similarly the other two derivatives are related by

$$T^i_{j;k} = T^i_{j,k} + \frac{1}{n}\{T^i_k \overset{1}{\Gamma}{}^l_{lj} + Y_k T^i_m \Delta^m \overset{1}{\Gamma}{}^l_{jl} - T^m_j \delta^i_k \overset{1}{\Gamma}{}^l_{ml}\} \tag{4.17}$$

and

$$T^i_{j\|k} = T^i_{j,k} + \frac{1}{n}\{T^i_k \overset{2}{\Gamma}{}^l_{lj} + Y_k T^i_m \Delta^m \overset{2}{\Gamma}{}^l_{jl} - T^m_j \delta^i_k \overset{2}{\Gamma}{}^l_{ml}\}. \tag{4.18}$$

From equation (4.15) one can easily obtain the following commutation formulae

$$2T^i_{j|[kh]} = T^r_j \overset{1}{Q}{}^{*i}_{rkh} - T^i_r \overset{2}{Q}{}^{*r}_{jkh} - T^i_{j|r} \overset{2}{S}{}^{*r}_{kh}. \tag{4.19}$$

where

$$\overset{1}{Q}{}^{*i}_{jkh} \overset{\text{def}}{=} 2\{\partial_{[h}\overset{1}{\Gamma}{}^{*i}_{|j|k]}+(\Delta_m\overset{1}{\Gamma}{}^{*i}_{j[k})\partial_{h]}X^m+\overset{1}{\Gamma}{}^{*m}_{j[k}\overset{1}{\Gamma}{}^{*i}_{|m|h]}\}, \tag{4.20}$$

and

$$\overset{2}{Q}{}^{*i}_{jkh} \overset{\text{def}}{=} 2\{\partial_{[h}\overset{2}{\Gamma}{}^{*i}_{|j|k]}+(\Delta^m\overset{2}{\Gamma}{}^{*i}_{j[k})\partial_{h]}Y_m+\overset{2}{\Gamma}{}^{*m}_{j[k}\overset{2}{\Gamma}{}^{*i}_{|m|h]}\} \tag{4.21}$$

and $\overset{2}{S}{}^{*l}_{kh}=2\overset{2}{\Gamma}{}^{*l}_{[kh]}$.

Substituting from (4.7) in (4.20) one obtains on simplification

$$\overset{1}{Q}{}^{*i}_{jkh} = \overset{1}{R}{}^{i}_{jkh}+\overset{1}{\Gamma}{}^{l}_{lj}\overset{1}{S}{}^{i}_{hk}-2\delta^{i}_{[k}\{\partial_{h]}\overset{1}{\Gamma}{}^{l}_{lj}+(\Delta_m\overset{1}{\Gamma}{}^{l}_{lj})\partial_{h]}X^m-\overset{1}{\Gamma}{}^{l}_{lm}\overset{1}{\Gamma}{}^{m}_{|j|h]}+\overset{1}{\Gamma}{}^{l}_{lj}\overset{1}{\Gamma}{}^{p}_{|p|h]}\}. \tag{4.22}$$

Similarly from (4.10) and (4.21) we obtain

$$\begin{aligned}\overset{2}{Q}{}^{*i}_{jkh} = {} & \overset{2}{R}{}^{i}_{jkh}-2\delta^{i}_{[k}\{\partial_{h]}\overset{1}{\Gamma}{}^{l}_{lj}+(\Delta^m\overset{1}{\Gamma}{}^{l}_{lj})\partial_{h]}Y_m+\overset{2}{\Gamma}{}^{m}_{|j|h]}\overset{1}{\Gamma}{}^{p}_{pm}-\overset{1}{\Gamma}{}^{p}_{|p|h]}\overset{1}{\Gamma}{}^{l}_{lj}-\overset{1}{\Gamma}{}^{p}_{pm}Y_{h]}\Delta^m\overset{1}{\Gamma}{}^{l}_{lj}\}- \\ & -2\{\partial_{[h}(Y_k\Delta^i\overset{1}{\Gamma}{}^{l}_{lj})+\partial_{[h}Y_m\Delta^m(Y_{k]}\Delta^i\overset{1}{\Gamma}{}^{l}_{jl})+\overset{1}{\Gamma}{}^{l}_{lj}Y_{[h}\Delta^i\overset{1}{\Gamma}{}^{p}_{|p|k]}-\overset{2}{\Gamma}{}^{m}_{j[k}Y_{h]}\Delta^i\overset{1}{\Gamma}{}^{p}_{pm}- \\ & -Y_{[k}(\Delta^m\overset{1}{\Gamma}{}^{l}_{lj})\overset{2}{\Gamma}{}^{i}_{|m|h]}\}+\overset{2}{S}{}^{i}_{kh}\overset{1}{\Gamma}{}^{l}_{lj}.\end{aligned} \tag{4.23}$$

Remarks. i) In analogy with (4.20) and (4.21) we can also define $'\overset{1}{Q}{}^{i}_{jkh}$, $'\overset{2}{Q}{}^{i}_{jkh}$ and $\overset{1}{\tilde{Q}}{}^{i}_{jkh}$ and $\overset{2}{\tilde{Q}}{}^{i}_{jkh}$ by replacing the coefficient of connection $\overset{1}{\Gamma}{}^{*i}_{jk}$ by $'\overset{1}{\Gamma}{}^{i}_{jk}$ and $\overset{1}{\Gamma}{}^{i}_{jk}$ respectively. One can observe that all the six entities defined by equations analogous to (4.20) and (4.21) are projectively invariant entities.

ii) By taking two different types of covariant derivatives one can easily establish certain commutation formulae, as well as some other projective invariants can be defined.

If we contract equation (3.5) for i and h we get

$$\Phi_k = \frac{1}{2(n-1)}\{\overset{1}{\bar{R}}{}^{l}_{kl}-\overset{1}{R}{}^{l}_{kl}+2P\overset{1}{S}{}^{l}_{kl}\}. \tag{4.24}$$

Also it is easy to observe that

$$\overset{1}{\bar{\Gamma}}{}^{r}_{r}(\delta^{i}_{h}\overset{1}{\bar{S}}{}^{l}_{kl}-\delta^{i}_{k}\overset{1}{\bar{S}}{}^{l}_{hl}-(n-1)\overset{1}{\bar{S}}{}^{i}_{kh}) = \overset{1}{\Gamma}{}^{r}_{r}(\delta^{i}_{h}\overset{1}{S}{}^{l}_{kl}-\delta^{i}_{k}\overset{1}{S}{}^{l}_{hl}-(n-1)\overset{1}{S}{}^{i}_{kh}), \tag{4.25}$$

therefore substituting from (4.24) in (3.5) and using the value of P from (4.1) we define the following invariant entity:

$$\overset{1}{W}{}^{i}_{kh} \overset{\text{def}}{=} \overset{1}{R}{}^{i}_{kh}-\frac{2}{(n-1)}\delta^{i}_{[h}\left(\overset{1}{R}{}^{l}_{k]l}-\frac{1}{n}\overset{1}{\Gamma}{}^{r}_{r}\overset{1}{S}{}^{l}_{k]l}\right)-\frac{1}{n}\overset{1}{\Gamma}{}^{r}_{r}\overset{1}{S}{}^{i}_{kh}. \tag{4.26}$$

From (4.26) it is easily observed that $\overset{1}{W}{}^{i}_{ki}=0$.

Differentiating equation (4.26) with respect to X^j and using (2.5) we define the following invariant entity:

$$\overset{1}{W}{}^i_{jkh} \overset{\text{def}}{=\!=} \overset{1}{R}{}^i_{jkh} - \frac{2}{(n-1)} \{\delta^i_{[h} \overset{1}{R}{}^l_{|j|k]l} - (\Delta_j \overset{1}{\Gamma}{}^l_{m[k}) X^m_{,l]} \delta^i_h - \tag{4.27}$$

$$- \Delta_j \overset{1}{\Gamma}{}^l_{m[h} X^m_{,l]} \delta^i_k\} + \frac{2}{n(n-1)} \delta^i_{[h} (\Delta_j \overset{1}{S}{}^l_{k]l} \overset{1}{\Gamma}{}^r_r) - 2(\Delta_j \overset{1}{\Gamma}{}^i_{m[k}) X^m_{,h]} - \frac{1}{n} \Delta_j (\overset{1}{\Gamma}{}^r_r \overset{1}{S}{}^i_{kh}).$$

Remark. $\overset{1}{W}{}^i_{jkh}$ is a tensor analogous to Weyl's projective curvature tensor in a Finsler space. Here one can easily establish the relationship between these two curvature tensors and also study the properties of $\overset{1}{W}{}^i_{jkh}$ analogous to Weyl's projective curvature tensor.

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