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On relative connectedness I.

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In their discussion of certain optimization problems, ST. SIMONS (1990) and S. HORVÁTH (1991) are using the following definition of a connected set: "a subset $K \subset X$ of a topological space $(X, \mathcal{T})$ is connected, if the relations $K \subset A \cup B$ and $K \cap A \cap B = \emptyset$ imply $K \subset A$ or $K \subset B$, where $A, B \in \mathcal{T}$."

This definition of connectedness has been generalized by H. KÖNIG in his conference "The Topological Minimax Theorem "(Debrecen University, Institute of Mathematics and Informatics, 12 October 1995). Let X be a nonvoid set and $\mathcal{S}$ a family of subset of X. H. König defines connectedness with respect to the family of sets $\mathcal{S}$ as follows: "The subset $K \subset X$ is connected with respect to the family of sets $\mathcal{S}$, if for any sets $A, B \in \mathcal{S}$ the relations $K \subset A \cup B$ and $K \cap A \cap B = \emptyset$ imply $K \subset A$ or $K \subset B$. "

Remarks. 1. For $\mathcal{S} = \mathcal{T}$ we get back the definition used by Simons, i.e. the classical notion of connectedness.

2. If $\mathcal{S}_1 \subset \mathcal{S}_2$ and $K \subset X$ is connected with respect to the family of sets $\mathcal{S}_2$, then K is connected with respect to the family of sets $\mathcal{S}_1$ too.

3. If for $K \subset X$ there do not exist sets $A, B \in \mathcal{S}$ for wich $K \subset A \cup B$ then K is connected with respect to $\mathcal{S}$.

The last remark motivates the following

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Definition 1. Let X be an arbitrary set and $\mathcal{S}$ a given family of subsets. The subset $K \subset X$ is $(n-1)$-connected with respect to $\mathcal{S}$, if for any sets $A_1, A_2, \dots, A_n \in \mathcal{S}$ the relations $K \subset \bigcup_{i=1}^{n} A_i$ and $K \cap A_{i_0} \cap \Big(\bigcup_{\substack{i=1 \\ i \neq i_0}}^{n} A_i \Big) = \emptyset$ imply $K \subset A_{i_0}$ or $K \subset \bigcup_{\substack{i=1 \\ i \neq i_0}}^{n} A_i$.

Remarks. 4. The notion of 1-connected set coincides with that of connected set in the sense of König. The notion just introduced can also be called chain-connectedness.

5. If $\mathcal{S}_1, \subset \mathcal{S}$ and $K \subset X$ is $(n-1)$-connected with respect to $\mathcal{S}$, then K is $(n-1)$-connected also with respect to S_1.

6. The following examples will show that the notion just introduced essentially depends on both the family of sets $\mathcal{S}$, and the natural number $n \in \mathbb{N}$.

Let $X = \mathbb{R}$ and let $\mathcal{S}$ be the set of those open intervals the length of which is not greater than 1, and let $K = (0,2) \cup (2,3) \cup \{10\}$. One sees that by Remark 3 K is a 1-connected set. Similarly, K is 2-connected and 3-connected too. It can be shown that K is not 4-connected. Indeed, let $A = (0,1)$, $B = (1/2, 3/2)$, $C = (1,2)$, $D = (2,3)$, $E = (19/2, 21/2)$. One verifies that the relations $K \subset A \cup B \cup C \cup D \cup E$ and $K \cap E \cap (A \cup B \cup C \cup D) = \emptyset$ are satisfied, but neither $K \subset E$ nor $K \subset A \cup B \cup C \cup D$ holds.

7. The notion introduced proves helpful in studying classic connectedness in a topological space $(X, \mathcal{T})$ by investigating k-connectedness with respect to a topological base $\mathcal{B}$. This will become clear in the sequel.

Theorem 1. *If $K \subset X$ is an $(n-1)$-connected set with respect to the family of sets $\mathcal{S}$ and $n \geq 3$, then K is also $(n-2)$-connected with respect to $\mathcal{S}$, hence it is connected with respect to $\mathcal{S}$.*

Proof. We suppose that K is $(n-1)$-connected with respect to the family of sets $\mathcal{S}$. Let the sets $A_1, A_2, \dots, A_{n-1} \in \mathcal{S}$ satisfy $K \subset \bigcup_{i=1}^{n-1} A_i$ and $K \cap A_{i_0} \cap \Big(\bigcup_{\substack{i=1 \\ i \neq i_0}}^{n-1} A_i \Big) = \emptyset$. Moreover, let $i_1 \neq i_0$, $1 \leq i_1 \leq n-1$ and $A_n = A_{i_1}$. It follows that we also have $K \subset \bigcup_{i=1}^{n} A_i$ and $K \cap A_{i_0} \cap \Big(\bigcup_{\substack{i=1 \\ i \neq i_0}}^{n} A_i \Big) = \emptyset$. Since K is $(n-1)$-connected with respect to the family of sets $\mathcal{S}$, we see that either $K \subset A_{i_0}$ or $K \subset \bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{n} A_i = \bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{n-1} A_i$ holds, i.e. K is an $(n-2)$-connected set with respect to the family of sets $\mathcal{S}$.

The above examples show that the converse of the theorem is not true.

A family of sets $\mathcal{S}$ is said to be closed with respect to finite unions, if $A, B \in \mathcal{S}$ always implies $A \cup B \in \mathcal{S}$.

Theorem 2. *If the family of sets $\mathcal{S}$ is closed with respect to finite unions and if $K \subset X$ is $(n-1)$-connected with respect to the family of sets $\mathcal{S}$, then K is $(k-1)$-connected with respect to the family of sets $\mathcal{S}$ for any natural number $k \geq 2$.*

Proof. Suppose that $K \subset X$ is an $(n-1)$-connected set with respect to $\mathcal{S}$. It follows that K is 1-connected with respect to $\mathcal{S}$. Let the sets $A_1, A_2, \ldots, A_k \in \mathcal{S}$ satisfy the relations $K \subset \bigcup\limits_{i=1}^{k} A_i$ and $K \cap A_{i_0} \cap \left(\bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{k} A_i\right) = \emptyset$. (If no such sets $A_1, A_2, \ldots, A_k \in \mathcal{S}$ exist then K is trivially $(k-1)$-connected.) Let $B_1 = A_{i_0}$ and $B_2 = \bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{k} A_i$. Let the conditions $K \subset B_1 \cup B_2$ and $K \cap B_1 \cap B_2 = \emptyset$ be satisfied. Since K is connected with respect to the family of sets $\mathcal{S}$ and $B_1, B_2 \in \mathcal{S}$ it follows that either $K \subset B_1 = A_{i_0}$ or $K \subset B_2 = \bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{k} A_i$ i.e. K is $(k-1)$-connected with respect to $\mathcal{S}$.

Theorem 3. *If $K \subset X$ is $(n-1)$-connected with respect to $\mathcal{S}$ and the conditions $K \cap A_j \cap \left(\bigcup\limits_{\substack{i=1\\ i\neq j}}^{n} A_i\right) = \emptyset$ and $K \subset \bigcup\limits_{i=1}^{n} A_i$ are satisfied for any natural number j $(1 \leq j \leq n)$, then there exists a natural number i_0 $(1 \leq i_0 \leq n)$ such that $K \subset A_{i_0}$.*

Proof. On the basis of the given conditions we have either $K \subset A_j$ or $K \subset \bigcup\limits_{\substack{i=1\\ i\neq j}}^{n} A_i$ for any natural number $j = 1, 2, \ldots, n$. Let us suppose that there does not exist any $i_0 \in \mathbb{N}$, $1 \leq i_0 \leq n$ for which $K \subset A_{i_0}$. Then, for any natural number $j = 1, 2, \ldots, n$ we obtain $K \subset \bigcup\limits_{\substack{i=1\\ i\neq j}}^{n} A_i$. Hence the conditions $K \cap A_j \subset \left(\bigcup\limits_{\substack{i=1\\ i\neq j}}^{n} A_i\right) \cap A_j$, $j = 1, 2, \ldots, n$ are satisfied, i.e. $K \cap A_j \subset$

$\left(\bigcup\limits_{\substack{i=1\\ i\neq j}}^{n} A_i\right) \cap A_j \cap K,\ j = 1,2,\dots,n$. In view of $K \cap A_j \cap \left(\bigcup\limits_{\substack{i=1\\ i\neq j}}^{n} A_i\right) = \emptyset,\ j = 1,2,\dots,n$ we have $K \cap A_j = \emptyset,\ j = 1,2,\dots,n$. From this we get $K \cap \left(\bigcup\limits_{i=1}^{n} A_i\right) = \emptyset$. Now $K \subset \bigcup\limits_{i=1}^{n} A_i$ implies $K = \emptyset$ and so $K \subset A_{i_0}$ holds for any natural number i_0, $1 \le i_0 \le n$ which contradicts our hypothesis. Thus the theorem is proved.

In what follows, let $\mathcal{S}$ be a given family of sets satisfying $\emptyset \in \mathcal{S}$. Let $\mathcal{S}^n$ denote the family of sets

$$\mathcal{S}^1(\cup)\mathcal{S}^{n-1} = \{A \cup B \mid A \in \mathcal{S}^1,\ B \in \mathcal{S}^{n-1}\}$$

where $\mathcal{S}^1 = \mathcal{S}$, and let $\mathcal{S}^{\cup} = \bigcup\limits_{n=1}^{\infty} \mathcal{S}^n$. One sees that $\mathcal{S}^{\cup}$ is closed with respect to finite unions and

$$\mathcal{S} = \mathcal{S}^1 \subset \mathcal{S}^2 \subset \dots \subset \mathcal{S}^n \subset \dots .$$

Theorem 4. *If $K \subset X$ is connected (1-connected) with respect to $\mathcal{S}^n$, then K is n-connected with respect to $\mathcal{S}$.*

Proof. Let us suppose that K is connected with respect to the family of sets $\mathcal{S}^n$. Let the sets $A_1, A_2, \dots, A_{n+1} \in \mathcal{S}$ satisfy the conditions $K \subset \bigcup\limits_{i=1}^{n+1}$ and $K \cap A_{i_0} \cap \left(\bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{n+1} A_i\right) = \emptyset$ where $1 \le i_0 \le n+1$.

Since $A_{i_0}, \bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{n+1} A_i \in \mathcal{S}^n$ and K is connected with respect to $\mathcal{S}^n$, it follows that $K \subset A_{i_0}$ or else $K \subset \bigcup\limits_{\substack{i=1\\ i\neq i_0}}^{n+1} A_i$ and so we get that K is an n-connected set with respect to $\mathcal{S}$. The converse of the theorem does not hold.

Remark 8. If $\mathcal{S}$ is closed with respect to finite unions then $\mathcal{S} = \mathcal{S}^1 = \mathcal{S}^2 = \dots = \mathcal{S}^n = \dots$ and so $\mathcal{S} = \mathcal{S}^n = \mathcal{S}^{\cup}$, $\forall n \in \mathbb{N}$. Hence we get that the set $K \subset X$ is connected with respect to $\mathcal{S}$ if and ony if it is n-connected with respect to $\mathcal{S}$ for any natural number $n \in \mathbb{N}$, $n \ge 2$.

In what follows we are going to investigate certain properties of n-connectedness, taking into account those of classical connectedness. In order to supplement our notations, let us remark that the closure of a set $L \subset X$ with respect to a family of sets $\mathcal{S}$ will be the set $\widetilde{L} \subset X$, defined by $\widetilde{L} = \bigcap\limits_{\substack{S \in \mathcal{S}^{\cup}\\ L \subset S}} S$.

Remark 9. If $\widetilde{L}$ is the closure of the set L with respect to $\mathcal{S}$, then $\widetilde{L}$ the closure of L also with respect to $\mathcal{S}^n$ for any natural number $n \in \mathbb{N}$, and $\widetilde{L}$ is the closure of L with respect to $\mathcal{S}^{\cup}$ too.

Theorem 5. *If $K \subset X$ is an n-connected set with respect to $\mathcal{S}$ and $K \subset L \subset \widetilde{K}$ then L too is n-connected with respect to $\mathcal{S}$.*

Proof. Let $A_0, A_1, \dots, A_n \in \mathcal{S}$ be sets satisfying $L \subset \bigcup_{i=0}^{n} A_i$ and $L \cap A_{i_0} \cap \Big(\bigcup_{\substack{i=0\\ i\neq i_0}}^{n} A_i \Big) = \emptyset$. Now $K \subset L$ implies that $K \subset \bigcup_{i=0}^{n} A_i$ and $K \cap A_{i_0} \cap \Big(\bigcup_{\substack{i=0\\ i\neq i_0}}^{n} A_i \Big) = \emptyset$. Since K is n-connected with respect to $\mathcal{S}$, either $K \subset A_{i_0}$ or $K \subset \bigcup_{\substack{i=0\\ i\neq i_0}}^{n} A_i$. Now $A_{i_0}, \bigcup_{\substack{i=0\\ i\neq i_0}}^{n} A_i \in \mathcal{S}^{\cup}$ implies that $\widetilde{K} \subset A_{i_0}$ or $\widetilde{K} \subset \bigcup_{\substack{i=0\\ i\neq i_0}}^{n} A_i$, i.e. we get $L \subset A_{i_0}$ or $L \subset \bigcup_{\substack{i=0\\ i\neq i_0}}^{n} A_i$. Thus L is n-connected with respect to $\mathcal{S}$.

Theorem 6. *If the members of the family of sets $\{K_i \mid i \in I\}$ are n-connected with respect to $\mathcal{S}$ and $\bigcap_{i\in I} K_i \neq \emptyset$ then $K = \bigcup_{i\in I} K_i$ too is n-connected with respect to $\mathcal{S}$.*

Proof. Let $A_0, A_1, \dots, A_n \in \mathcal{S}$ be sets, such that $K \subset \bigcup_{j=0}^{n} A_j$ and $K \cap A_{j_0} \cap \Big(\bigcup_{\substack{j=0\\ j\neq j_0}}^{n} A_j \Big) = \emptyset$. It follows that for any $i \in I$ we also have $K_i \subset \bigcup_{j=0}^{n} A_j$ and $K_i \cap A_{j_0} \cap \Big(\bigcup_{\substack{j=0\\ j\neq j_0}}^{n} A_j \Big) = \emptyset$. Since K_i is n-connected with respect to $\mathcal{S}$ for any $i \in I$, it follows that $K_i \subset A_{j_0}$ or $K_i \subset \bigcup_{\substack{j=0\\ j\neq j_0}}^{n} A_j$ for any $i \in I$. Now let $I = I_1 \cup I_2$ with

$$I_1 = \{i \in I \mid K_i \subset A_{j_0}\},$$

$$I_2 = \{i \in I \mid K_i \subset \bigcup_{\substack{j=0\\ j\neq j_0}}^{n} A_j\}.$$

If $I_2 = \emptyset$ (or $I_1 = \emptyset$) then the theorem is true, because $K_i \subset A_{j_0}$ (or $K_i \subset \bigcup\limits_{\substack{j=0\\ j\neq j_0}}^{n} A_j$) for any $i \in I$, and consequently $K \subset A_{j_0}$ (or $K \subset \bigcup\limits_{\substack{j=0\\ j\neq j_0}}^{n} A_j$). Now suppose $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. Let $L_1 = \bigcup\limits_{i\in I_1} K_i$, $L_2 = \bigcup\limits_{i\in I_2} K_i$. One sees that $L_1 \cap L_2 \subset \Big(\bigcup\limits_{\substack{j=0\\ j\neq j_0}}^{n} A_j\Big) \cap A_{j_0}$ and $L_1 \cap L_2 \subset K$, hence $L_1 \cap L_2 \subset K \cap A_{j_0} \cap \Big(\bigcup\limits_{\substack{j=0\\ j\neq j_0}}^{n} A_j\Big)$ i.e. $L_1 \cap L_2 = \emptyset$. From $\bigcap\limits_{i\in I} K_i \subset K_i$, $\forall i \in I$ we infer that $\bigcap\limits_{i\in I} K_i \subset L_1$ and $\bigcap\limits_{i\in I} K_i \subset L_2$, hence $\bigcap\limits_{i\in I} K_i \subset L_1 \cap L_2$. Thus we obtain $\bigcap\limits_{i\in I} K_i = \emptyset$ and this contradicts the conditions of the theorem.

Theorem 7. *If for any two points $x, y \in X$ there exists a set $K_{xy} \subset X$ which is n-connected with respect to $\mathcal{S}$ and satisfies $x, y \in K_{xy}$ then the space X is n-connected with respect to $\mathcal{S}$.*

Proof. Let $x \in X$ be a fixed point and $y \in X$ a variable point whose range is the whole space. For any point $y \in X$ let K_{xy} be the set, n-connected with respect to $\mathcal{S}$, the existence of which is postulated in the theorem. Since $\bigcap\limits_{y\in X} K_{xy} \supset \{x\} \neq \emptyset$ the previous theorem implies that $X = \bigcup\limits_{y\in X} K_{xy}$ is an n-connected set with respect to $\mathcal{S}$.

Remark 10. Let X be an arbitrary set and $\mathcal{S}$ an arbitrary family of subset of X. For any natural number $n \in \mathbb{N}$ and any point $x \in X$ the sets $K_0 = \emptyset$ and $K_x = \{x\}$ are n-connected with respect to $\mathcal{S}$.

In what follows, let X and Y be two arbitrary sets and $\mathcal{S} \subset \mathcal{P}(X)$, $\mathcal{R} \subset \mathcal{P}(Y)$ two given families of sets, where $\mathcal{P}(U) = \{A \subset U\}$.

Theorem 8. *If X is n-connected with respect to $\mathcal{S}$ and if there exists a function $f : X \to Y$ such that $\bigcup\limits_{R\in\mathcal{R}} R = f(X)$ and $f^{-1}(R) \in \mathcal{S}$ for any $R \in \mathcal{R}$ then Y is n-connected with respect to $\mathcal{R}$.*

Proof. It will be sufficient to prove that $f(X) \subset Y$ is n-connected with respect to $\mathcal{R}$. Indeed, if $f(X) \neq Y$ is n-connected then Y too is n-connected (since it has no covering by elements of $\mathcal{R}$), and whenever $Y \supset Z \supset f(X)$ holds, Z is also n-connected.

On the basis of this it suffices to prove the theorem for a surjective function.

Let now be $R_0, R_1, \ldots, R_n \in \mathcal{R}$ and we suppose that $Y \subset \bigcup\limits_{i=0}^{n} R_i$ and $Y \cap R_{i_0} \cap \Big(\bigcup\limits_{\substack{i=0\\ i\neq i_0}}^{n} R_i \Big) = \emptyset$. We put $S_i = f^{-1}(R_i) \in \mathcal{S}$, $i = 0, 1, \ldots, n$ and we get $X \subset \bigcup\limits_{i=0}^{n} S_i$ and $X \cap S_{i_0} \cap \Big(\bigcup\limits_{\substack{i=0\\ i\neq i_0}}^{n} S_i \Big) = \emptyset$. Since X is an n-connected space with respect to $\mathcal{S}$, it follows that $X \subset S_{i_0}$ or $X \subset \bigcup\limits_{\substack{i=0\\ i\neq i_0}}^{n} S_i$ hence $Y = f(X) \subset f(S_{i_0}) = R_{i_0}$ or $Y = f(X) \subset f\Big(\bigcup\limits_{\substack{i=0\\ i\neq i_0}}^{n} S_i \Big) = \bigcup\limits_{\substack{i=0\\ i\neq i_0}}^{n} R_i$. Thus the space Y is n-connected with respect to $\mathcal{R}$.

In what follows, let $\{X_i \mid i \in I\}$ be a given family of sets, and for any $i \in I$ let $\mathcal{S}_i \subset \mathcal{P}(X_i)$ be given. Let $X = \prod\limits_{i\in I} X_i$ denote the cartesian product of the family of sets, and let

$$\mathcal{S} = \left\{ \prod_{i\in I} A_i \mid A_i \in \mathcal{S}_i \cup \{X_i\},\ |\{i \in I_j,\ A_i \neq X_i\}| \in \mathbb{N} \right\}.$$

One sees that $x \in I$.

Theorem 9. *If the space X_i is n-connected with respect to $\mathcal{S}_i$ for any $i \in I$, then X is n-connected with respect to $\mathcal{S}$.*

Proof. For any $i \in I$, let us fix a point $x_i \in X_i$. Let $x = (x_i) \in \prod\limits_{i\in I} X_i$ and let us denote by C the set of those points $y \in \prod\limits_{i\in I} X_i$ which have only finitely many of their coordinates different from the corresponding coordinate of x. If $A \in \mathcal{S}^{\cup}$ then it is easy to see that $C \subset A$ if and only if $A = X$. Let $\widetilde{C}$ denote the closure of the set C with respect to the family of sets $\mathcal{S}^{\cup}$. There follows that $\widetilde{C} = X$. By Theorem 6 it will be sufficient to prove that C is n-connected with respect to $\mathcal{S}^{\cup}$, i.e. with respect to $\mathcal{S}$. By Theorem 7 it suffices to prove that for any point $y \in C$ there exists a subset $C_y \subset C$ such that $x, y \in C_y$ and which is n-connected with respect to $\mathcal{S}$. Therefore let $y = (y_i) \in C$. There exist indices $i_1, i_2, \ldots, i_n \in I$ such that $x_i = y_i, i \in I \setminus \{i_1, \ldots, i_n\}$. For any natural number $k \leq n$ let

$$B_k = \left\{ z = (z_i) \in \prod_{i\in I} X_i \;\middle|\; \begin{matrix} z_{i_e} = x_{i_\ell}, & 1 \leq \ell < k; & z_{i_k} \in X_{i_k} \\ z_{i_e} = y_{i_\ell}, & k < \ell \leq n; & z_i = x_i,\ i \in I \setminus \{i_i, \ldots, i_n\} \end{matrix} \right\}.$$

It follows that $y \in B_1$, $x \in B_n$, $B_k \cap B_{k+1} \neq \emptyset$, $k = 1, 2, \dots, n-1$ (indeed let $z_i = x_i$, $i = i_\ell$, $\ell \leq k$ and for $i \in I \setminus \{i_1, \dots, i_n\}$, let $z_i = y_i$, $i = i_\ell$, $k+1 \leq \ell \leq n$. With these notations we obtain $z = (z_i) \in B_k \cap B_{k+1}$).

There exist bijective mappings $h_k : X_{i_k} \to B_k$, $k = 1, 2, \dots, n$. Let $\mathcal{S}'_k = \mathcal{S} \cap \mathcal{P}(B_k)$, $k = 1, 2, \dots, n$. It can be shown that for any set $L \in \mathcal{S}'_k$ one has $h_k^{-1}(L) \in \mathcal{S}_{i_k}$, $k = 1, 2, \dots, n$ and consequently $B_k = h_k(X_{i_k})$ is n-connected with respect to $\mathcal{S}'_k$. It can easily be shown that the sets B_k are n-connected with respect to $\mathcal{S}$ too. This implies that all the sets B_1, $B_1 \cup B_2, \dots, \bigcup_{k=1}^{n} B_k$ are n-connected with respect to $\mathcal{S}$ hence $C_y = \bigcup_{k=1}^{n} B_k$ is the set we wanted to obtain. This completes the proof of the theorem.

In what follows, we are going to determine the connected components of the space X.

Definition 2. The set $C \subset X$ is an n-connected component with respect to $\mathcal{S}$ of the space X, if for any set $C_1 \subset X$, n-connected with respect to $\mathcal{S}$, $C \subset C_1$ implies $C = C_1$.

Theorem 10. *Any two noncoinciding components, n-connected with respect to $\mathcal{S}$, are disjoint.*

Proof. Let K_1,K_2 be two components, n-connected with respect to $\mathcal{S}$, for wich $K_1 \cap K_2 \neq \emptyset$. Thus $K_1 \cup K_2$ is also an n-connected set with respect to $\mathcal{S}$. The inclusions $K_1 \subset K_1 \cup K_2$ and $K_2 \subset K_1 \cup K_2$ imply $K_1 = K_1 \cup K_2 = K_2$, and the theorem is proved.

Theorem 11. *If the set $K \subset X$ is n-connected with respect to $\mathcal{S}$, then the space has a component $C \subset X$, n-connected with respect to $\mathcal{S}$, such that $K \subset C$.*

Proof. Let C be a component, n-connected with respect to $\mathcal{S}$, for wich $C \cap K \neq \emptyset$. It follows that $C \cup K$ too is n-connected with respect to $\mathcal{S}$, hence $C = C \cup K$ i.e. $K \subset C$. (C is the union of those sets, n-connected with respect to $\mathcal{S}$, which have nonvoid intersection with K.)

Theorem 12. *The space X can be represented as the union of its components, n-connected with respect to $\mathcal{S}$.*

Proof. Let $x \in X$ be an arbitrary point. The fact that the one-point set $\{x\}$ is n-connected with respect to $\mathcal{S}$, implies the existence of a component C_x such that $x \in C_x$. Hence $X = \bigcup_{x \in X} C_x$ is the desired representation.

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