On the general solution of rectangle-type functional equations

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0. *Introduction*

Several papers have dealt with certain functional equations which, considering their geometric meaning, have been given the name of geometric figures (for example [2], [5], [6], [9]).

The functional equation

$$f(x+y, u+v)+f(x+y, u-v)+f(x-y, u+v)+f(x-y, u-v) = 4f(x, u), \tag{1}$$

where $f\colon \mathbf{R}^2\to\mathbf{R}$ and $x, y, u, v\in\mathbf{R}$ is said to be the rectangle-equation.

Generalizing (1) O. Em. GHEORGHIU [4] dealt among others with the following rectangle-type functional equations:

$$\begin{cases} F(x+y, u+v)+F(x+y, u-v)+F(x-y, u+v)+F(x-y, u-v) = \\ = 4F(x, u)+F\left(\dfrac{x+y}{2x}, \dfrac{u+v}{2u}\right)+F\left(\dfrac{x+y}{2x}, \dfrac{u-v}{2u}\right)+ \\ +F\left(\dfrac{x-y}{2x}, \dfrac{u+v}{2u}\right)+F\left(\dfrac{x-y}{2x}, \dfrac{u-v}{2u}\right), \end{cases} \tag{2}$$

where $F\colon \mathbf{R}_+^2\to\mathbf{R}$ and (2) holds for all $x, u\in\mathbf{R}_+$, $y, v\in\mathbf{R}$ with $|y|<x$, $|v|<u$. Here $\mathbf{R}_+$ is the set of positive real numbers.

$$\begin{cases} G(x+y, u+v)+G(x+y, u-v)+G(x-y, u+v)+G(x-y, u-v) = \\ = 4G\left(\sqrt{x^2+y^2}, \sqrt{u^2+v^2}\right)+G\left(\dfrac{x+y}{m\sqrt{x^2+y^2}}, \dfrac{u+v}{p\sqrt{u^2+v^2}}\right)+ \\ +G\left(\dfrac{x+y}{m\sqrt{x^2+y^2}}, \dfrac{u-v}{p\sqrt{u^2+v^2}}\right)+G\left(\dfrac{x-y}{m\sqrt{x^2+y^2}}, \dfrac{u+v}{p\sqrt{u^2+v^2}}\right)+ \\ +G\left(\dfrac{x-y}{m\sqrt{x^2+y^2}}, \dfrac{u-v}{p\sqrt{u^2+v^2}}\right), \end{cases} \tag{3}$$

where $G\colon \mathbf{R}_+^2\to\mathbf{R}$ and (3) holds for all $x, u\in\mathbf{R}_+$, $y, v\in\mathbf{R}$ with $|y|<x$, $|v|<u$, and $m, p\in\mathbf{R}_+$ are arbitrary constants.

He determined the general measurable solutions of (2) and (3).

In this paper we find the generai solutions of some generalizations of (2) and (3) and we also determine the solutions under certain regularity conditions.

1. *Notations and preliminary results*

We shall use the following notations and results of paper [8].

For $\underline{x}=(x_1, \ldots, x_n)$, $\underline{\lambda}=(\lambda_1, \ldots, \lambda_n)\in \mathbf{R}^n_+$ let

$$\underline{\lambda}\underline{x} = (\lambda_1 x_1, \ldots, \lambda_n x_n).$$

If $\underline{x}\in \mathbf{R}^n_+$ then $\underline{x}_i(t)$ will denote the vector

$$(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n).$$

The function $H\colon \mathbf{R}^n_+ \to \mathbf{R}$ is said to be a *generalized Jensen-function* if

$$2H\left(\underline{x}_i\left(\frac{t+s}{2}\right)\right) = H(\underline{x}_i(t))+H(\underline{x}_i(s)) \tag{4}$$

holds for every $i=1, \ldots, n$, $\underline{x}\in \mathbf{R}^n_+$ and $t, s\in \mathbf{R}_+$.

The class of generalized Jensen-functions will be denoted by $J(\mathbf{R}^n_+ \to \mathbf{R})$.

Let $F\colon \mathbf{R}^n_+ \to \mathbf{R}$ be an arbitrary function and $\Delta_{\underline{\lambda}} F\colon \mathbf{R}^n_+ \to \mathbf{R}$ be the function defined by

$$\Delta_{\underline{\lambda}} F(\underline{x}) = F(\underline{\lambda}\underline{x}) - F(\underline{x}), \quad \underline{x}, \underline{\lambda}\in \mathbf{R}^n_+ . \tag{5}$$

In the paper [8] the following result was proved:

Theorem 1. (see theorem 3.1 in [8]). *Let $F\colon \mathbf{R}^n_+ \to \mathbf{R}$ be a function, such that the function $\Delta_{\underline{\lambda}} F\colon \mathbf{R}^n_+ \to \mathbf{R}$ defined by* (5) *belongs to $J(\mathbf{R}^n_+ \to \mathbf{R})$ for all $\underline{\lambda}\in \mathbf{R}^n_+$. Then there exist homomorphisms $m_i\colon \mathbf{R}_+ \to \mathbf{R}$ $(i=1, \ldots, n)$, multiadditive functions $A^k_l\colon \mathbf{R}^l \to \mathbf{R}$ $\left(i=1, \ldots, n;\ k=0, \ldots, \binom{n}{l};\ l=1, \ldots, n-1\right)$ and an $A_0\in \mathbf{R}$ constant, such that*

$$\begin{cases} F(\underline{x}) = A^0_n(x_1, \ldots, x_n)+A^1_{n-1}(x_1, \ldots, x_{n-1})+\ldots+A^n_{n-1}(x_2, \ldots, x_n)+ \\ +A^1_{n-2}(x_1, \ldots, x_{n-2})+\ldots+A^{\frac{n(n-1)}{2}}_{n-2}(x_3, \ldots, x_n)+\ldots+ \\ +A^1_2(x_1, x_2)+\ldots+A^{\frac{n(n-1)}{2}}_2(x_{n-1}, x_n)+A^1_1(x_1)+\ldots+ \\ +A^n_1(x_n)+\sum_{i=1}^n m_i(x_i)+A_0 \end{cases} \tag{6}$$

holds for all $\underline{x}\in \mathbf{R}^n_+$.

2. *Generalization of equation* (2)

Let $F\colon \mathbf{R}^n_+ \to \mathbf{R}$ be an arbitrary function and let T^{t_i} $(i=1, \ldots, n)$ be a linear transformation defined by

$$T^{t_i}F(\underline{x}) = F(\underline{x}_i(t_i)), \quad \underline{x}\in \mathbf{R}^n_+, \ t_i\in \mathbf{R}_+ . \tag{7}$$

Let us denote by $\prod_{i=1}^n T^{t_i}$ the product of transformations $T^{t_1}, \ldots, T^{t_n}$.

The functional equation

$$(8)\qquad \begin{cases} \prod_{i=1}^{n} (T^{t_i+s_i}+T^{t_i-s_i})\,F(\underline{x}) = 2^n \prod_{i=1}^{n} T^{t_i}F(\underline{x})+ \\ +\prod_{i=1}^{n} \left(T^{\frac{t_i+s_i}{2t_i}}+T^{\frac{t_i-s_i}{2t_i}}\right)F(\underline{x}), \end{cases}$$

where $\underline{t}=(t_1, \dots, t_n)\in \mathbf{R}_+^n$, $\underline{s}=(s_1, \dots, s_n)\in \mathbf{R}^n$ with $|s_i|<t_i$ $(i=1, \dots, n)$ is a generalization of (2) for functions $F\colon \mathbf{R}_+^n \to \mathbf{R}$, since for $n=2$ equation (8) reduces to (2).

The following result gives the general solution of the functional equation (8).

Theorem 2. *If the function* $F\colon \mathbf{R}_+^n \to \mathbf{R}$ *satisfies the functional equation* (8) *for all* $\underline{t}\in \mathbf{R}_+^n$, $\underline{s}\in \mathbf{R}^n$ *with* $|s_i|<t_i$ $(i=1, 1, \dots, n)$, *then for all* $\underline{x}\in \mathbf{R}_+^n$ *has the form* (6), *where the functions*

$$A_l^k\colon \mathbf{R}^l \to \mathbf{R} \quad \left(i = 1, \dots, n;\ k = 0, \dots, \binom{n}{l};\ l = 1, \dots, n-1\right)$$

are multiadditive, $m_i\colon \mathbf{R}_+ \to \mathbf{R}$ $(i=1, \dots, n)$ *are homomorphisms and*

$$(9)\qquad A_0 = -\left[A_n^0\left(\frac{1}{2}, \dots, \frac{1}{2}\right) + \dots + A_1^1\left(\frac{1}{2}\right) + \dots + A_1^n\left(\frac{1}{2}\right) + \sum_{i=1}^{n} m_i\left(\frac{1}{2}\right)\right]$$

is a constant.

Proof. By the substitution

$$(10)\qquad t_i+s_i = x_i,\quad t_i-s_i = y_i \quad (i = 1, \dots, n)$$

we obtain from (8) the functional equation

$$(11)\qquad \begin{cases} \prod_{i=1}^{n} (T^{x_i}+T^{y_i})\,F(\underline{x}) = 2^n \prod_{i=1}^{n} T^{\frac{x_i+y_i}{2}} F(\underline{x})+ \\ +\prod_{i=1}^{n} \left(T^{\frac{2x_i}{x_i+y_i}}+T^{\frac{2y_i}{x_i+y_i}}\right)F(\underline{x}) \end{cases}$$

on the domain of transformation (10), i.e. (11) holds for all $x_i, y_i\in \mathbf{R}_+$ $(i=1, \dots, n)$.

Let $\underline{\lambda}=(\lambda_1, \dots, \lambda_n)\in \mathbf{R}_+^n$, then we get from (11) the functional equation

$$\prod_{i=1}^{n} (T^{\lambda_i x_i}+T^{\lambda_i y_i})\,F(\underline{x}) = 2^n \prod_{i=1}^{n} T^{\lambda_i\frac{x_i+y_i}{2}} F(\underline{x})+$$
$$+\prod_{i=1}^{n} \left(T^{\frac{2x_i}{x_i+y_i}}+T^{\frac{2y_i}{x_i+y_i}}\right)F(\underline{x}).$$

Subtracting this equation from (11) we see that the function $\Delta_{\underline{\lambda}}F\colon \mathbf{R}_+^n \to \mathbf{R}$ defined by (5) satisfies the functional equation

$$\prod_{i=1}^{n} (T^{x_i}+T^{y_i})\,\Delta_{\underline{\lambda}}F(\underline{x}) = 2^n \prod_{i=1}^{n} T^{\frac{x_i+y_i}{2}} \Delta_{\underline{\lambda}}F(\underline{x})$$

for all $x_i, y_i\in \mathbf{R}_+$ $(i=1, \dots, n)$, $\underline{\lambda}\in \mathbf{R}_+^n$, which implies that $\Delta_{\underline{\lambda}}F$ belongs to $J(\mathbf{R}_+^n \to \mathbf{R})$. Thus the function $F\colon \mathbf{R}_+^n \to \mathbf{R}$ satisfies the conditions of theorem 1., therefore there exist multiadditive functions $A_l^k\colon \mathbf{R}^l \to \mathbf{R}$ $(i=1, \ldots, n;\ k=0, \ldots, \binom{n}{l};\ l=1, \ldots, n-1)$, homomorphisms $m_i\colon \mathbf{R}_+ \to \mathbf{R}$ $(i=1, \ldots, n)$ and constant $A_0 \in \mathbf{R}$, such that F is of the form (6) for all $\underline{x} \in \mathbf{R}_+^n$.

It is easy to see that the function (6) satisfies the functional equation (8) if A_0 has the form (9). ∎

For $n=2$ we get

Corollary 1. *If the function* $F\colon \mathbf{R}_+^2 \to \mathbf{R}$ *satisfies the functional equation* (2) *for all* $x, u \in \mathbf{R}_+$; $y, v \in \mathbf{R}$ *with* $|y| < x$, $|v| < u$, then

$$F(x, y) = A(x, y) + a_1(x) + a_2(y) + m_1(x) + m_2(y) + a_0 \tag{12}$$

for all $(x, y) \in \mathbf{R}_+^2$, *where* $A\colon \mathbf{R}^2 \to \mathbf{R}$ *is a biadditive function,* $a_1, a_2\colon \mathbf{R} \to \mathbf{R}$ *are additive functions,* $m_1, m_2\colon \mathbf{R}_+ \to \mathbf{R}$ *are homomorphisms and*

$$a_0 = -A\left(\frac{1}{2}, \frac{1}{2}\right) - a_1\left(\frac{1}{2}\right) - a_2\left(\frac{1}{2}\right) - m_1\left(\frac{1}{2}\right) - m_2\left(\frac{1}{2}\right)$$

is a constant.

This corollary is a simple generalization of Theorem 1. in [4] and gives the solutions of (2) under certain regularity conditions as follows:

Corollary 2. *Let* $F\colon \mathbf{R}_+^2 \to \mathbf{R}$ *be an unknown function, which satisfies the functional equation* (2) *for all* $x, u \in \mathbf{R}_+$, $y, v \in \mathbf{R}$ *with* $|y| < x$, $|v| < u$, *then the following conditions are equivalent:*

a) *There exists* $(x_0, y_0) \in \mathbf{R}_+^2$, *such that F is continuous at the points* (x_0, y_0), $(2x_0, y_0)$, $(x_0, 2y_0)$;

b) *The function* F *is measurable on* $\mathbf{R}_+^2$.

Further from any of these conditions it follows that

$$F(x, y) = A(4xy-1) + a(2x-1) + b(2y-1) + c \ln 2x + d \ln 2y \tag{13}$$

for all $(x, y) \in \mathbf{R}_+^2$, *where* $A, a, b, c, d \in \mathbf{R}$ *are arbitrary constants.*

Proof. It is sufficient to show that F is of the form (13) in all these cases.

If the function F satisfies the functional equation (2), then by Corollary 1. F has the form (12). Thus it is easy to see that

$$F(2x, y) - F(x, 2y) = a_1(x) - a_2(y) + m_1(2) + m_2(2), \tag{14}$$

$$2F(x, y) - F(2x, y) = a_2(y) + m_1(x) + m_2(y) + a_0 - m_1(2), \tag{15}$$

$$F(2x, y) - F(x, y) = A(x, y) + a_1(x) + m_1(2) \tag{16}$$

are valid for all $(x, y) \in \mathbf{R}_+^2$.

1) If F satisfies a), then (14) implies that the additive functions $a_1, a_2\colon \mathbf{R} \to \mathbf{R}$ are continuous at the points x_0 and y_0 respectively hence (see [1])

$$a_1(x) = c_1 x, \quad a_2(y) = c_2 y \quad (x, y \in \mathbf{R}), \tag{17}$$

where $c_1, c_2 \in \mathbf{R}$ are arbitrary constants.

Further, using the continuity of a_2 at the point y_0, we obtain from (15) that the homomorphisms m_1 and m_2 are also continuous at the points x_0 and y_0 respectively, thus we have (see [1])

$$m_1(x) = d_1 \ln x, \quad m_2(y) = d_2 \ln y \quad (x, y \in \mathbf{R}_+), \tag{18}$$

where $d_1, d_2 \in \mathbf{R}$ are arbitrary constants.

Finally the continuity of a_1 at x_0 and formula (16) implies that the biadditive function A is continuous at the point $(x_0, y_0) \in \mathbf{R}^2$, therefore (see [1])

$$A(x, y) = Axy \quad ((x, y) \in \mathbf{R}^2), \tag{19}$$

where $A \in \mathbf{R}$ is an arbitrary constant.

(17), (18), (19) and the formula (12) show that (13) is true provided that the condition a) is satisfied.

2) If condition b) holds, then by a method similar to the previous one we can prove the measurability of the functions a_1, a_2, m_1, m_2 and A, thus (17), (18) and (19) are valid in this case too (see [1]). Hence F is of the form (13).

This completes the proof of Corollary 2.

3. *Generalization of equation* (3)

Let $G: \mathbf{R}^n_+ \to \mathbf{R}$ be an arbitrary function. By the help of transformation T^{t_i} defined by (7) we obtain the following generalization of functional equation (3):

$$\begin{cases} \prod_{i=1}^{n} (T^{t_i+s_i} + T^{t_i-s_i}) G(\underline{x}) = 2^n \prod_{i=1}^{n} T^{\sqrt{t_i^2+s_i^2}} G(\underline{x}) + \\ + \prod_{i=1}^{n} \left(T^{p_i \sqrt{t_i^2+s_i^2}}^{\frac{t_i+s_i}{}} + T^{p_i\sqrt{t_i^2+s_i^2}}^{\frac{t_i-s_i}{}}\right) G(\underline{x}), \end{cases} \tag{20}$$

where $t_i \in \mathbf{R}_+$, $s_i \in \mathbf{R}$ with $|s_i| < t_i$ $(i=1, \ldots, n)$ and $p_i \in \mathbf{R}_+$ $(i=1, \ldots, n)$ are arbitrary constants.

For $n=2$, (20) gives the functional equation (3) for the function $G: \mathbf{R}^2_+ \to \mathbf{R}$.

We are going to prove the following

Theorem 3. *Let $G: \mathbf{R}^n_+ \to \mathbf{R}$ be an arbitrary function, which satisfies the functional equation* (20) *for all $t_i \in \mathbf{R}_+$, $s_i \in \mathbf{R}$ with $|s_i| < t_i$ $(i=1, \ldots, n)$, where $p_i \in \mathbf{R}_+$ $(i=1, \ldots, n)$ are arbitrary constants. Then*

$$\begin{cases} G(\underline{x}) = A_n^0(x_1^2, \ldots, x_n^2) + A_{n-1}^1(x_1^2, \ldots, x_{n-1}^2) + \ldots + A_{n-1}^n(x_2^2, \ldots, x_n^2) + \\ + A_{n-2}^1(x_1^2, \ldots, x_{n-2}^2) + \ldots + A_{n-2}^{\frac{n(n-1)}{2}}(x_3^2, \ldots, x_n^2) + \ldots + A_2^1(x_1^2, x_2^2) + \ldots + \\ + A_2^{\frac{n(n-1)}{2}}(x_{n-1}^2, x_n^2) + A_1^1(x_1^2) + \ldots + A_1^n(x_n^2) + \sum_{i=1}^{n} m_i(x_i^2) + A_0 \end{cases} \tag{21}$$

for all $\underline{x} \in \mathbf{R}^n_+$, *where the functions* $A_i^k: \mathbf{R}^i \to \mathbf{R}$ $\left(i=1, \ldots, n;\ k=0, \ldots, \binom{n}{l}\right.$;

$l=1, \dots, n-1$) *are multiadditive,* $m_i: \mathbf{R}_+ \to \mathbf{R}$ $(i=1, \dots, n)$ *are homomorphisms and*

$$(22)\qquad A_0 = -\left[A_n^0\left(\frac{1}{p_1^2}, \dots, \frac{1}{p_n^2}\right) + \dots + A_1^1\left(\frac{1}{p_1^2}\right) + \dots + A_1^n\left(\frac{1}{p_n^2}\right) + \sum_{i=1}^n m_i\left(\frac{1}{p_i^2}\right)\right]$$

is a constant.

Proof. By the substitutions

$$(23)\qquad t_i+s_i = \sqrt{x_i}, \quad t_i-s_i = \sqrt{y_i}$$

and

$$(24)\qquad H(x_1, \dots, x_n) = G(\sqrt{x_1}, \dots, \sqrt{x_n}) \quad (\underline{x} = (x_1, \dots, x_n) \in \mathbf{R}_n^+)$$

we obtain from (20) the functional equation

$$(25)\qquad \begin{cases} \prod_{i=1}^n (T^{x_i}+T^{y_i}) H(\underline{x}) = 2^n \prod_{i=1}^n T^{\frac{x_i+y_i}{2}} H(\underline{x}) + \\ + \prod_{i=1}^n \left(T^{\frac{2x_i}{p_i^2(x_i+y_i)}} + T^{\frac{2y_i}{p_i^2(x_i+y_i)}}\right) H(\underline{x}) \end{cases}$$

on the domain of transformation (23), i.e. for all $x_i, y_i \in \mathbf{R}_+$ $(i=1, \dots, n)$.

Let $\underline{\lambda}=(\lambda_1, \dots, \lambda_n) \in \mathbf{R}_+^n$, then (25) gives the functional equation

$$\prod_{i=1}^n (T^{\lambda_i x_i}+T^{\lambda_i y_i}) H(\underline{x}) = 2^n \prod_{i=1}^n T^{\lambda_i \frac{x_i+y_i}{2}} H(\underline{x}) +$$

$$+ \prod_{i=1}^n \left(T^{\frac{2x_i}{p_i^2(x_i+y_i)}} + T^{\frac{2y_i}{p_i^2(x_i+y_i)}}\right) H(\underline{x}).$$

Subtracting this equation from (25) we obtain that the function $\Delta_{\underline{\lambda}} H: \mathbf{R}_+^n \to \mathbf{R}$ defined by (5) satisfies the functional equation

$$\prod_{i=1}^n (T^{x_i}+T^{y_i}) \Delta_{\underline{\lambda}} H(\underline{x}) = 2^n \prod_{i=1}^n T^{\frac{x_i+y_i}{2}} \Delta_{\underline{\lambda}} H(\underline{x})$$

for all $x_i, y_i \in \mathbf{R}_+$ $(i=1, \dots, n)$, $\underline{\lambda} \in \mathbf{R}_+^n$, which easily implies that $\Delta_{\underline{\lambda}} H$ belongs to $J(\mathbf{R}_+^n \to \mathbf{R})$ for all fixed $\underline{\lambda} \in \mathbf{R}_+^n$. Thus, we obtain that H is of the form (6) too. From (24) we get the formula (21) for the function G. It is easy to see that the function (21) satisfies the functional equation (20) if A_0 has the form (22).

For $n=2$ then we have the following

Corollary 3. *If the function* $G: \mathbf{R}_+^2 \to \mathbf{R}$ *satisfies* (3) *for all* $x, u \in \mathbf{R}_+$, $y, v \in \mathbf{R}$ *with* $|y|<x$, $|v|<u$, *where* $m, p \in \mathbf{R}_+$ *are arbitrary constants, then*

$$(26)\qquad G(x, y) = A(x^2, y^2) + a_1(x^2) + a_2(y^2) + m_1(x^2) + m_2(y^2) + a_0$$

for all $(x, y)\in \mathbf{R}_+^2$, *where the function* $A: \mathbf{R}^2\to\mathbf{R}$ *is biadditive,* $a_1, a_2: \mathbf{R}\to\mathbf{R}$ *are additive,* $m_1, m_2: \mathbf{R}_+\to\mathbf{R}$ *are homomorphisms and*

$$a_0 = -A\left(\frac{1}{m^2}, \frac{1}{p^2}\right) - a_1\left(\frac{1}{m^2}\right) - a_2\left(\frac{1}{p^2}\right) - m_1\left(\frac{1}{m^2}\right) - m_2\left(\frac{1}{p^2}\right) \tag{27}$$

is a constant.

Corollary 3. is a generalization of Theorem 2. of [4]. We can easily prove that the solution of (3) under certain regularity conditions (e.g. measurability, continuity) is

$$\begin{cases} G(x, y) = A_0 \ln (mx)^2 + A_1 \ln (py)^2 + B_0(1-m^2x^2) + \\ + B_1(1-p^2y^2) + B_2(1-m^2x^2)(1-p^2y^2), \quad (x, y)\in\mathbf{R}_+^2, \end{cases} \tag{27}$$

where $A_0, A_1, B_0, B_1, B_2\in\mathbf{R}$ are arbitrary constants.

(27) corresponds to formula (23) of [4].

4. *Remarks*

1) It is possible to investigate the functional equations (8) and (20) under regularity condition b) and a modification of a).

2) The functional equations (8), (20) can be investigated for functions of type $F: P^n\to A$ or $G: P^n\to A$ respectively, where P denotes the positive elements of an ordered field and A is an abelian group in which every equation $2^k x = y$ $(k=1, \ldots, n)$ has an unique solution for x.

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