Super geodesic congruence in a subspace of a Finsler space

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Summary. AMUR [1] defined that the hypergeodesic curve is characterised by the property that the union curvarure vector is orthogonal to the first curvature vector of the curve with respect to the hypersurface V_m of the Riemannian space V_n. This concept was introduced in a Finsler hypersurface by PRASAD [2]. Further the special and hyperasymptotic congruences of a Finsler subspace were defined and studied by one of the authors [3, 4]. In view of the definition of hypergeodesic curves, we consider in the present article that the supergeodesic congruence is characterised by the property that the special curvature vector is orthogonal to the geodesic curvature vector of the congruence in the Finsler subspace F_m, and study some of its properties.

1. *Introduction.* Let a subspace $F_m, x^i=x^i(s)$, $i=1, 2 \dots n$ $\alpha=1, 2 \dots m$ be immersed in an n-dimensional Finsler space F_n. Consider a curve $C: x^i=x^i(s)$; of the subspace, s being its arc length. The components $x'^i=\dfrac{dx^i}{ds}$ and $u'^\alpha=\dfrac{du^\alpha}{ds}$ of the unit tangent vector to C are related by $x'^i=B^i_\alpha u'^\alpha$, where $B^i_\alpha=\dfrac{\partial x^i}{\partial u^\alpha}$. A line element (u, u') is thus determined at a point of C. All the quantities in our discussion are considered for this line element.

The metric tensors $g_{\alpha\beta}(u, u')$ and $g_{ij}(x, x')$ of F_m and F_n respectively are related by

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x')B^i_\alpha B^j_\beta \tag{1.1}$$

There exists a set of vectors $n^{*i}_{(\sigma)}(x, x')$, $\sigma=m+1, \dots, n$, normal to the subspace and are called secondary normals. These are given by the solutions [7].

$$g_{ij}(x, x')n^{*i}_{(\sigma)}B^j_\alpha = n^*_{(\sigma)j}B^j_\alpha = 0. \tag{1.2}$$

$$g_{ij}(x, x')n^{*i}_{(\sigma)}n^{*j}_{(\nu)} = \delta^\nu_\sigma \psi_{(\nu)} \quad \text{(no summation on } \nu) \tag{1.3}$$

and

$$g_{ij}(x, n^*_{(\sigma)})\dot{n}^i_{(\sigma)}n^{*j}_{(\sigma)} = 1. \tag{1.4}$$

Let a set of $(n-m)$ linearly independent vectors $\underset{(\sigma)}{\mu^i}(x, x')$, $\sigma=m+1, \dots, n$ define $(n-m)$ congruences of curves which are such that exactly one curve of each congruence passes through each point of the space F_n and so through each point of F_m. At a point P of the subspace, we write

$$\underset{(\sigma)}{\mu^i} = l^\alpha_{(\sigma)}(u, u')B^i_\alpha + \sum_\nu \Gamma_{(\sigma\nu)}(u, u')n^{*i}_{(\nu)}. \tag{1.5}$$

Suppose that the vectors $\underset{(\sigma)}{\mu^i}$ with m linearly independent vectors of F_m form a set of n linearly independent vectors in F_m which is possible if $|\Gamma_{(\sigma\nu)}|\neq 0$.

These vectors $\underset{(\sigma)}{\mu^i}$ satisfy the equations

$$(1.6) \qquad g_{ij}(x, x')\underset{(\sigma)}{\mu^i}\underset{(\sigma)}{\mu^j} = 1.$$

Using (1.5) in (1.6) and simplifying in view of equations (1.2) and (1.3), we get

$$(1.7) \qquad g_{\alpha\beta}(u, u')l^\alpha_{(\sigma)}l^\beta_{(\sigma)} = 1-\sum_\nu \Gamma^2_{(\sigma\nu)}\psi_{(\nu)}$$

Consider the contravariant components $\lambda^i(x, x')$ of a congruence of curves which is not necessarily a member of the set of congruences defined by (1.5). At a point of the subspace, it may be expressed as

$$(1.8) \qquad \lambda^i = t^\alpha B^i_\alpha+\sum_\nu C_{(\nu)}n^{*i}_{(\nu)}$$

and satisfies

$$(1.9) \qquad g_{ij}(x, x')\lambda^i\lambda^j = 1.$$

The covariant derivative of (1.6) with respect to u^β in the direction of C is given by [5].

$$(1.10) \qquad \frac{\delta\lambda^i}{\delta s} = W^\alpha B^i_\alpha+\sum_\nu D_{(\nu)}n^{*i}_{(\nu)}$$

where

$$(1.11) \qquad W^\alpha = \frac{\delta t^\alpha}{\delta s}+\sum_\nu C_{(\nu)}A^\alpha_{(\nu)\beta}u'^\beta$$

and

$$(1.12) \qquad D_{(\nu)} = \Omega^*_{(\nu)\alpha\beta}(u, u')t^\alpha u'^\beta+\frac{\delta C_{(\nu)}}{\delta s}+\sum_\sigma C_{(\sigma)}N^{(\sigma)}_{(\nu)\beta}u'^\beta.$$

The quantities $\Omega^*_{(\nu)\alpha\beta}$ are called secondary second fundamental tensors and $A^\alpha_{(\nu)\beta}$ and $N_{(\nu)\beta}$ are defined in [5].

Definition (1.1) The scalar defined by

$$(1.13) \qquad K^2_{N^*} = g_{ij}(x, x')\Big(\sum_\nu D_{(\nu)}n^{*i}_{(\nu)}\Big)\Big(\sum_\sigma (D_{(\sigma)}n^{*j}_{(\sigma)}\Big)$$

is called secondary normal curvature of the congruence λ^i in the subspace F_m [6]. With the help of equations (1.3), the above expression can be written as

$$(1.14) \qquad K^2_{N^*} = \sum_\nu \psi_{(\nu)}D^2_{(\nu)}.$$

A direction along which the secondary normal curvature of the congruence λ^i in the subspace vanishes is called the asymptotic direction and a curve whose direction at each point of it is asymptotic is called an asymptotic line of the congruence in the subspace [6].

Definition (1.2). The quantity K_G, defined by

$$(1.15) \qquad K^2_G = g_{\alpha\beta}(u, u')W^\alpha W^\beta$$

is called the geodesic curvature of the congruence λ^i along C in F_m [6]. If the geodesic curvature of the congruence λ^i in the subspace vanishes at every point of the curve C, the curve is called a λ-geodesic [6].

Definition (1.3). The scalar K, defined by

$$K^2 = g_{ij}(x, x')\left(\frac{\delta\lambda^i}{\delta s}\right)\left(\frac{\delta\lambda^j}{\delta s}\right) \tag{1.16}$$

is called the absolute curvature of the congruence in the subspace [6]. If the absolute curvature of the congruence vanishes along a curve C, the curve is called the absolute geodesic of the congruence in the subspace [6].

With the help of equations (1.10), (1.14) and (1.15), equation (1.16) may be written as

$$K^2 = K_G^2 + K_{N^*}^2. \tag{1.17}$$

2. *Hyper asymptotic and supergeodesic congruences*

Definition (2.1). The scalar $K_{(\sigma)H}$ defined by

$$K_{(\sigma)H} \overset{\text{def}}{=} g_{ij}(x, x') \underset{(\sigma)}{\mu^i} \frac{\delta\lambda^j}{\delta s} \tag{2.1}$$

is called hyperasymptotic curvature of the congruence λ^i on F_m [4]. A congruence λ^i is said to be hyperasymptotic congruence relative to the congruence $\underset{(\sigma)}{\mu^i}$ if at each point of a curve of F_m, $K_{(\sigma)H}=0$ [4]. Its differential equation, therefore is given by

$$g_{\alpha\beta}\, l^{\alpha}_{(\sigma)} W^{\beta} + \sum_{\nu} \Gamma_{(\sigma\nu)} \psi_{(\nu)} D_{(\nu)} = 0 \tag{2.2}$$

where we have used equations (1.5) and (1.10) in (2.1).

Definition (2.2). A congruence λ^i is said to be a special congruence with respect to a curve C in F_m such that the surface determined by the geodesic curvature vectors of the congruence λ^i with respect to F_n and F_m at each point of the curve C contains the congruence $\underset{(\sigma)}{\mu^i}$ [3]. By its definition we have

$$\underset{(\sigma)}{\mu^i} = A_{(\sigma)} W^{\alpha} B^i_{\alpha} + B_{(\sigma)} \frac{\delta\lambda^i}{\delta s} \tag{2.3}$$

where the vectors W^{α} and $\frac{\delta\lambda^i}{\delta s}$ are respectively the geodesic curvature vectors of λ^i with respect to F_m and F_n and $A_{(\sigma)}$ and $B_{(\sigma)}$ are parameters to be determined.

Using equations (1.5), and (1.10) in equation (2.3) we have

$$l^{\alpha}_{(\sigma)} B^i_{\alpha} + \sum_{\nu} \Gamma_{(\sigma\nu)} n^{*i}_{(\nu)} = A_{(\sigma)} W^{\alpha} B^i_{\alpha} + B_{(\sigma)}(W^{\alpha} B^i_{\alpha} + D_{(\nu)} n^{*i}_{(\nu)}). \tag{2.4}$$

Since $n^{*i}_{(\nu)}$ and B^i_α are linearly independent, we have

$$l^\alpha_{(\sigma)} = A_{(\sigma)} + B_{(\sigma)})W^\alpha \tag{2.5}$$

and

$$\frac{1}{B_{(\sigma)}} = \frac{D_{(\nu)}}{\Gamma_{(\sigma\nu)}}. \tag{2.6}$$

These equations (2.5) and (2.6) give

$$A_{(\sigma)} = \left[\frac{l^\alpha_{(\sigma)}}{W^\alpha} - \frac{\Gamma_{(\sigma\nu)}}{D_{(\nu)}}\right]. \tag{2.7}$$

Multiplying (2.5) by $g_{\alpha\beta} l^\beta_{(\sigma)}$ and using (1.7), we get

$$\left(1 - \sum_\nu \Gamma^2_{(\sigma\nu)} \psi_{(\nu)}\right) = (A_{(\sigma)} + B_{(\sigma)}) g_{\alpha\beta} W^\alpha l^\beta_{(\sigma)}. \tag{2.8}$$

Eliminating $A_{(\sigma)}$ and $B_{(\sigma)}$ from (2.5) and (2.8) we get

$$W^\alpha - \left(1 - \sum_\nu \Gamma^2_{(\sigma\nu)} \psi_{(\nu)}\right)^{-1} g_{\beta\gamma} W^\beta l^\gamma_{(\sigma)} l^\alpha_{(\sigma)} = 0 \tag{2.9}$$

Let us suppose that the congruence $\underset{(\sigma)}{\mu^i}$ be not normal to the subspace, the solutions of the system of m-differential equations (2.9) determine the special congruence of F_m with respect to $\underset{(\sigma)}{\mu^i}$ [3].

The vector with contravariant component T^α, called the special curvature vector of the congruence λ^i relative to $\underset{(\sigma)}{\mu^i}$, is given by

$$T^\alpha = W^\alpha - \left(1 - \sum_\nu \Gamma^2_{(\sigma\nu)} \psi_{(\nu)}\right)^{-1} g_{\beta\gamma} W^\beta l^\gamma_{(\sigma)} l^\alpha_{(\sigma)}. \tag{2.10}$$

The magnitude of the special curvature vector given by

$$K^2_T = g_{\alpha\beta}(u, u') T^\alpha T^\beta \tag{2.11}$$

is called the special curvature of the congruence λ^i [3].

Using (1.15) and (2.10) in (2.11) we get

$$K^2_T = K^2_G - \frac{2(g_{\alpha\beta} W^\alpha l^\beta_{(\sigma)})^2}{\left(1 - \sum_\nu \Gamma^2_{(\sigma\nu)} \psi_{(\nu)}\right)} + \frac{g_{\alpha\beta} l^\alpha_{(\sigma)} l^\beta_{(\sigma)} (g_{\gamma\delta} W^\gamma l^\delta_{(\sigma)})^2}{\left(1 - \sum_\nu \Gamma^2_{(\sigma\nu)} \psi_{(\nu)}\right)^2}. \tag{2.12}$$

From (2.12), we have the following

Theorem (2.1). *The special and geodesic curvatures of the congruence λ^i, are identical if the vectors $l^\alpha_{(\sigma)}$, $\sigma = m+1, \ldots, n$, are orthogonal to the vector W^α.*

Proof. If the vector $l^\alpha_{(\sigma)}$ be perpendicular to the vector W^α, then we have

$$g_{\alpha\beta}(u, u') l^\alpha_{(\sigma)} W^\beta = 0.$$

Using this result in (2.12), we get the statement. We shall now define the super-geodesic congruence.

Let $W^\alpha/K_G=L^\alpha$, then multiplying (2.10) by $g_{\alpha\beta}(u,u')L^\beta$, we have

$$(2.13)\quad g_{\alpha\beta}(u,u')T^\alpha L^\beta = K_G-\big(g_{\gamma\delta}(u,u')W^\gamma l^\delta_{(\sigma)}\big)\big(g_{\alpha\beta}l^\alpha_{(\sigma)}L^\beta\big)\big(1-\sum_\nu \Gamma^2_{(\sigma\nu)}{}'\psi_{(\nu)}\big)^{-1}.$$

Definition (2.2). The scalar $\overline{K}_S$, defined by

$$(2.14)\qquad \overline{K}_S = K_G-\big(g_{\alpha\beta}l^\alpha_{(\sigma)}L^\beta\big)\big(g_{\gamma\delta}W^\gamma l^\delta_{(\sigma)}\big)\big(1-\sum_\nu \Gamma^2_{(\sigma\nu)}\psi_{(\nu)}\big)^{-1}$$

be called the supergeodesic curvature of the cdngruence in F_m. If $\overline{K}_S$ vanishes along a curve C in F_m, the congruence relative to $\underset{(\sigma)}{\mu^i}$ is called a supergeodesic with respect to C. The differential equations of the supergeodesics are given by

$$(2.15)\qquad g_{\alpha\beta}(u,u')T^\alpha L^\beta = 0.$$

Theorem (2.2). *The supergeodesic and geodesic curvatures of the congruence are equal if the vector* $l^\alpha_{(\sigma)}$, $\sigma=m+1,\dots,n$, *is orthogonal to the vector* W^α.

PROOF. If the vector $l^\alpha_{(\sigma)}$ is orthogonal to the vector W^α, then we have

$$g_{\alpha\beta}(u,u')l^\alpha_{(\sigma)}W^\beta = 0$$

Using this result in the equation (2.14), we get

$$\overline{K}_s = K_G.$$

This was to be shown.

From theorems (2.1) and (2.2), we have

Theorem (2.3). *At a point of the subspace, the supergeodesic and the special curvatures of the congruence are identical if the vector* W^α *is orthogonal to the vector* $l^\alpha_{(\sigma)}$, $\sigma=m+1,\dots,n$ *each being equal to the geodesic curvature of the congruence.*

Theorem (2.4). *A supergeodesic congruence is characterised by the property that the special curvature vector is perpendicular to the geodesic curvature vector of the congruence* λ^i *in* F_m.

PROOF. The proof follows from the equation (2.15).

Now multiplying equations (2.2) and (2.14), by $D_{(\nu)}$ and K_G respectively and in the resulting equation using equation (1.17) we get

$$(2.16)\ K_{(\sigma)H}D_{(\nu)}+\overline{K}_S K_G = K^2+\frac{\dot{D}_{(\sigma)}}{\Gamma_{(\sigma\nu)}}g_{\alpha\beta}l^\alpha_{(\sigma)}W^\beta-\big(g_{\alpha\beta}l^\alpha_{(\sigma)}W^\beta\big)^2\big(1-\sum_\nu \Gamma^2_{(\sigma\nu)}\psi_{(\nu)}\big)^{-1}.$$

From (2.16) we may have the following

Theorem (2.5). *The absolute curvature of the congruence* λ^i *in* F_n *with respect to the hyperasymptotic line of the congruence* λ^i, *is the geometric mean of the supergeodesic and geodesic curvatures of the congruence if the vector* $l^\alpha_{(\sigma)}$ *be perpendicular to the vector* W^α *for* $\sigma=m+1,\dots,n$.

PROOF. Since $l^\alpha_{(\sigma)}$ is perpendicular to the vector W^α, we have

$$g_{\alpha\beta}l^\alpha_{(\sigma)}W^\beta = 0,\quad \text{for}\quad \sigma=m+1,\dots,n$$

Also

$$K_{(\sigma)H} = 0.$$

Using these results in (2.16), we have

$$K^2 = \overline{K}_S K_G.$$

This proves the theorem.

Theorem (2.6). *The absolute curvature of the congruence λ^i in F_n with respect to a supergeodesic on F_m is the geometric mean of the hyperasymptotic curvature of the congruence and the scalar $D_{(\nu)}$, if the vector $l^\alpha_{(\sigma)}$ be perpendicular to the vector W^α for $\sigma = m+1, \ldots, n$.*

PROOF. For a supergeodesic, we have

$$\overline{K}_S = 0$$

and $l^\alpha_{(\sigma)}$ being perpendicular to the vector W^α, gives

$$g_{\alpha\beta} l^\alpha_{(\sigma)} W'^\beta = 0, \quad \text{for} \quad \sigma = m+1, \ldots, n.$$

With the help of these results equation (2.16) reduces to

$$K^2 = K_{(\sigma)H} D_{(\nu)}.$$

This was to be shown.

Theorem (2.7). *The absolute curvature of the congruence λ^i in F_n is the geometric mean of the hyperasymptotic curvature and the scalar $D_{(\nu)}$ for a λ-geodesic.*

PROOF. For a λ-geodesic we have

$$W^\alpha = 0,$$

and consequently we have

$$K_G = 0.$$

In view of these results, equation (2.16) gives

$$K^2 = K_{(\sigma)H} D_{(\nu)}.$$

This proves the theorem.

3. *Geodesic curvature of the supergeodesic congruence*

Since the supergeodesic curvature of the supergeodesic congruence is zero, we have from equation (2.14) the following theorem.

Theorem (3.1). *The geodesic curvature of the supergeodesic congruence can be expressed in the form.*

$$K_G = \frac{g_{\alpha\beta} l^\alpha_{(\sigma)} W^\beta}{\sqrt{1 - \sum_\nu \Gamma^2_{(\sigma\nu)} \psi_{(\nu)}}}. \tag{3.1}$$

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