Multipliers of L^1-algebras with order convolution

By DAVID L. JOHNSON and CHARLES D. LAHR (Hanover, New Hampshire)

Abstract: For each $n, 1 \leqq n \leqq N$, let $L^1(a_n, b_n)$ be the commutative convolution measure algebra, under order convolution, of all Lebesgue integrable functions on the interval I_n of real numbers from a_n to b_n, where I_n is a topological semigroup under max multiplication. Then the multiplier algebra $\mathfrak{M}(L^1(a_n, b_n))$ of $L^1(a_n, b_n)$ is shown to be the Banach algebra obtained from $L^1(a_n, b_n)$ by the adjunction of an identity element. Moreover, the Banach space $L^1(\Pi I_n) = \hat{\bigotimes}_{n=1}^{N} L^1(a_n, b_n)$ of Lebesgue integrable functions on the product semigroup $\prod_{n=1}^{N} I_n$ becomes a commutative convolution measure algebra under order convolution, and it is shown that $\mathfrak{M}(\hat{\otimes} L^1(a_n, b_n)) = \hat{\otimes} \mathfrak{M}(L^1(a_n, b_n))$.

AMS(MOS) subject classification (1970). Primary 43A10, 43A20. Key words and phrases: convolution measure algebra, multiplier, tensor product.

1. *Introduction*

Let N be an arbitrary, but fixed, element of the positive integers $\mathbf{N}$ and, for each $n, 1 \leqq n \leqq N$, let $I_n = \{a_n, b_n\}$ be an interval of real numbers from a_n to b_n, where either a_n or b_n may possibly be infinite and I_n may or may not contain either a_n or b_n. Then, with the usual interval topology and max multiplication, each I_n is a locally compact idempotent commutative topological semigroup; with the product topology and componentwise multiplication, the product semigroup $S = \prod_{n=1}^{N} I_n$ has these same properties. Further, let $M(S)$ denote the Banach algebra under convolution product and total variation norm of all finite regular Borel measures on S. Then the Banach space $L^1(S)$ of all measures in $M(S)$ which are absolutely continuous with respect to (product) Lebesgue measure on S becomes a commutative semisimple Banach algebra in the inherited product $*$ which will be called order convolution. A multiplier of $L^1(S)$ is an automatically continuous linear operator $T: L^1(S) \to L^1(S)$ such that $T(f * g) = f * T(g)$, for all $f, g \in L^1(S)$. In this paper the multipliers of $L^1(S)$ are characterized. The result for the case of a single interval I_n, a special case of which has been obtained by Larsen using different methods [7], is easy to describe: the multiplier algebra $\mathfrak{M}(L^1(I_n))$ of $L^1(I_n)$ is the Banach algebra obtained from $L^1(I_n)$ by adjoining an identity. In the case where $N \geqq 2$, the multiplier algebra of $L^1(S)$ is the projective tensor product of the multiplier algebras $\mathfrak{M}(L^1(I_n)), 1 \leqq n \leqq N$.

The maximal ideal space $\Delta(L^1(I_n))$ of $L^1(I_n), I_n = \{a_n, b_n\}$, can be identified [6] with the interval $(a_n, b_n]$ (if $b_n = +\infty$, the right-end is compactified). For each $x \in (a_n, b_n]$, if $\varphi_x = K[a_n, x]$ is the characteristic function of $[a_n, x]$, then φ_x

determines a complex homomorphism via the pairing $\hat{f}(x)=\langle\varphi_x, f\rangle=\int_{a_n}^{x} f(t)\,dt$, $f\in L^1(I_n)$; correspondingly, $\Delta(L^1(S))=\prod_{n=1}^{N}(a_n, b_n]$ [1]. These and other facts about $L^1(I_n)=L^1(a_n, b_n)$ and $L^1(S)$ as Banach algebras can be found in [6, 1]. For results about tensor products, consult [4].

Now, $L^1(S)$ has the structure of a semisimple convolution measure algebra (CMA) in the sense of TAYLOR [8]; hence, there is an isometric imbedding $f\mapsto\nu_f$ of $L^1(S)$ into $M(\Gamma)$, where Γ is the structure semigroup of $L^1(S)$, and $\Delta(L^1(S))=\hat{\Gamma}$, the set of continuous semicharacters of Γ. Each $\chi\in\hat{\Gamma}$ determines a unique element of $\Delta(L^1(S))$ via the mapping $f\mapsto\int_\Gamma \chi\,d\nu_f$, $f\in L^1(S)$. Routine verification shows that the structure semigroup Γ_n of $L^1(I_n)$ consists of two ideal elements O_n and 1_n, and the pairs t^-, t^+, for all $t, a_n<t<b_n$. Further, $(\Gamma_n, <)$ is linearly ordered by specifying that

$$0_n < s^- < s^+ < t^- < t^+ < 1_n \quad \text{for all} \quad s, t, a_n < s < t < b_n.$$

The product on $\Gamma_n=[0_n, 1_n]$ is then max multiplication, and Γ_n has the compactified order topology (a one or two-point compactification when one of a_n or b_n, or both a_n and b_n, happen to be infinite). Moreover, $\hat{\Gamma}_n=\{\psi_x: x\in(a_n, b_n]\}$, where $\psi_x=K[0, x^+)$, $a_n<x<b_n$, and $\psi_{b_n}=K[0, b_n^+)=K(\Gamma_n)$. Also, because $L^1(S)=\hat{\bigotimes}_{n=1}^{N} L^1(I_n)$, the projective tensor product of the $L^1(I_n)$, $1\leqq n\leqq N$, the structure semigroup Γ of $L^1(S)$ is $\prod_{n=1}^{N}\Gamma_n$ [11].

The algebra of all multipliers of $L^1(S)$ is denoted by $\mathfrak{M}(L^1(S))$ and each $f\in L^1(S)$ determines a multiplier T_f by $T_f(g)=f*g$, $g\in L^1(S)$. Since $L^1(S)$ has an approximate identity bounded by 1, there is an isometric imbedding $T\mapsto\mu_T$ of $\mathfrak{M}(L^1(S))$ into $M(\Gamma)$ satisfying $\nu_{T(f)}=\mu_T*\nu_f$, $f\in L^1(S)$ [5, 12]. This realization of $\mathfrak{M}(L^1(S))$ will be utilized to show that $\mathfrak{M}(L^1(S))=\hat{\bigotimes}_{n=1}^{N}\mathfrak{M}(L^1(I_n))$ and that $\mathfrak{M}(L^1(I_n))$ is obtained from $L^1(I_n)$ by adjoining an identity. Section 2 is devoted to the case $N=1$ and contains the answer to a question posed by LARSEN [7, p. 244], while Section 3 is devoted to the general case. We remark that, for CMA's A, B, it is not in general true that $\mathfrak{M}(A\hat{\otimes}B)=\mathfrak{M}(A)\hat{\otimes}\mathfrak{M}(B)$. For example, let $A=B=L^1(\mathbf{R})$; then $\mathfrak{M}(L^1(\mathbf{R}))=M(\mathbf{R})$ and $\mathfrak{M}(L^1(\mathbf{R})\hat{\otimes}L^1(\mathbf{R}))=\mathfrak{M}(L^1(\mathbf{R}\times\mathbf{R}))=M(\mathbf{R}\times\mathbf{R})$ [10], while $M(\mathbf{R})\hat{\otimes}M(\mathbf{R})\subsetneqq M(\mathbf{R}\times\mathbf{R})$ [3, pp. 784—5].

2. *Multipliers of* $L^1(a, b)$

In this section, $N=1$ and we write $L^1(a, b)$ instead of $L^1(S)$. Let $A_\Gamma=\{\nu_f\in M(\Gamma): f\in L^1(a, b)\}$ and let δ_0 in $M(\Gamma)$ denote the Dirac measure of unit point mass concentrated at the identity 0 of $\Gamma=\Gamma_1$.

Theorem 1. *If* $T: L^1(a, b)\to L^1(a, b)$ *is a multiplier of* $L^1(a, b)$, *then* $\mu_T=c\delta_0+\nu_f$ *for some* c *in* $\mathbf{C}$; f *in* $L^1(a, b)$. *In other words,* $\mathfrak{M}(L^1(a, b))$ *is the Banach* *algebra obtained by adjoining the identity multiplier to the canonical image of* $L^1(a, b)$ *in* $\mathfrak{M}(L^1(a, b))$.

PROOF. Writing $\mu_T = c\delta_0 + \mu_T^+$, where $c = \mu_T(\{0\})$ (and so $\mu_T^+(\{0\}) = 0$), the result follows provided $\mu_T^+ \in A_\Gamma$. Since A_Γ is norm-closed in $M(\Gamma)$, it suffices to construct a sequence $\{\mu_k\}$ of measures in A_Γ which converges in norm to μ_T^+. We proceed now to the construction of just such a sequence.

Let $\{c_k\}$ be a strictly decreasing sequence of real numbers in (a, b) converging downward to a, and let $\{u_k\}$ be the approximate identity of $L^1(a, b)$ defined for each k by $u_k = (c_k - c_{k+1})^{-1} K[c_{k+1}, c_k]$. For each k, let $\omega_k = \nu_{Tu_k}$. Because $\nu_{Tf} = \mu_T * \nu_f$, for all $f \in L^1(a, b)$,

$$\langle \psi_x, \mu_T \rangle \langle \psi_x, \nu_f \rangle = \langle \psi_x, \mu_T * \nu_f \rangle = \langle \psi_x, \nu_{Tf} \rangle; \tag{1}$$

that is,

$$\mu_T[0, x^+) \int_a^x f(t)\,dt = \nu_{Tf}[0, x^+), \quad x \in (a, b],\ f \in L^1(a, b). \tag{2}$$

Setting $f = u_k$ in (2) and observing that $\int_a^x u_k(t)\,dt = 1$, for all x in $[c_k, b]$, yields

$$\mu_T[0, x^+) = \omega_k[0, x^+), \quad x \in [c_k, b]. \tag{3}$$

Further, since μ_T and ω_k are regular Borel measures on Γ, for all x in $(c_k, b]$,

$$\mu_T[c_k^+, x^+) = \omega_k[c_k^+, x^+), \tag{4}$$

with the convention that $[c_k^+, b^+) = [c_k^+, 1]$.

Now, for k in $\mathbf{N}$, define the measures μ_k and λ_k on Γ by setting $\mu_k = K[c_k^+, 1]\mu_T$ and $\lambda_k = K[c_k^+, 1]\omega_k$. Then, for each $\psi_x \in \hat{\Gamma}$, $x \in (a, b]$,

$$\langle \psi_x, \mu_k \rangle = \int_\Gamma \psi_x(s)\,d\mu_k(s) = \int_\Gamma K[0, x^+)(s) K[c_k^+, 1](s)\,d\mu_T(s) = \tag{5}$$

$$= \begin{cases} 0, & \text{if} \quad a < x \leqq c_k \\ \mu_T[c_k^+, x^+), & \text{if} \quad c_k < x \leqq b, \end{cases}$$

and, in like manner,

$$\langle \psi_x, \lambda_k \rangle = \begin{cases} 0, & \text{if} \quad a < x \leqq c_k \\ \omega_k[c_k^+, x^+), & \text{if} \quad c_k < x \leqq b. \end{cases} \tag{6}$$

Thus, in view of the identity (4), the measures μ_k and λ_k agree on $\hat{\Gamma}$; hence, $\mu_k = \lambda_k$. Consequently, $\mu_k \ll \omega_k$ and, since $\omega_k \in A_\Gamma$ and A_Γ is an L-subspace of $M(\Gamma)$, $\mu_k \in A_\Gamma$, for every k in $\mathbf{N}$.

The proof will be completed by setting $c = \mu_T(\{0\})$ and $\mu_T^+ = \mu_T - c\delta_0$, and proving that $\mu_T^+ = \lim_{k \to \infty} \mu_k$ in norm. Toward this end, observe that

$$\mu_T^+ - \mu_k = K(0, 1]\mu_T - K[c_k^+, 1]\mu_T = K(0, c_k^+)\mu_T, \tag{7}$$

so

$$\|\mu_T^+ - \mu_k\| = |K(0, c_k^+)\mu_T|(\Gamma) = |\mu_T|(0, c_k^+), \tag{8}$$

for each k. However, the interval $(0, c_1^+)$ is the countable disjoint union $\bigcup_{n=1}^{\infty} [c_{k+1}^+, c_k^+)$ and $|\mu_T|$ is a finite regular Borel measure on Γ, so

$$\sum_{k=1}^{\infty} |\mu_T|[c_{k+1}^+, c_k^+) = |\mu_T|(0, c_1^+) < +\infty. \tag{9}$$

Therefore, for every $\varepsilon > 0$, there exists an integer K such that

$$|\mu_T|(0, c_K^+) = \sum_{k=K}^{\infty} |\mu_T|[c_{k+1}^+, c_+^k) < \varepsilon. \tag{10}$$

This fact, together with equation (8), yields $\|\mu_T^+ - \mu_k\| < \varepsilon$, for all $k \geqq K$. ∎

LARSEN [7] obtained Theorem 1 for the special case $S = [0, 1]$ (i.e., $L^1(0, 1)$) using methods quite different from ours. Larsen also observed that, if a multiplier T of $L^1(0,1)$ is a compact operator on $L^1(0,1)$, then $T = T_f$, for some f in $L^1(0, 1)$. He stated the converse of this result as an open question, to which we now supply the answer.

Proposition 2. *For each positive integer n, the multiplier T_n of $L^1(0, 1)$ defined by the polynomial nx^{n-1} (i.e., $T_n f = (nx^{n-1}) * f$, for all f in $L^1(0, 1)$) is neither a compact nor even a weakly compact operator on $L^1(0, 1)$.*

PROOF. First, we exhibit a sequence $\{f_m\}_m$, contained in the unit ball of $L^1(0, 1)$, such that, for no n, does the sequence $\{T_n f_m\}_m$ have a norm-convergent subsequence. Let $f_1 \equiv 1$ and, for $m \geqq 2$, let $f_m = f_{m-1} * f_1$; then [6, MIDDLE, p. 4] $f_m(x) = mx^{m-1}$, for each m. Observe that $T_n f_m = f_n * f_m = f_{n+m}$, for all n, m in $\mathbf{N}$. Now, for each m, $f_m \geqq 0$ and $\|f_m\|_1 = \int_0^1 mx^{m-1} dx = 1$. Moreover, since $\lim_{m\to\infty} f_m(x) = 0$, for all x in $[0, 1)$, the only possible limit function in $L^1(0, 1)$ for a norm-convergent subsequence of the sequence $\{T_n f_m\}_m = \{f_{n+m}\}_m$ is the zero function. However, $\|f_{n+m}\|_1 = 1$, for all m, so every member of the sequence $\{T_n f_m\}_m$ is L^1-distance 1 from the zero function. Therefore, the norm-bounded sequence $\{f_m\}_m$ in $L^1(0, 1)$ is such that, for no n, does the sequence $\{T_n f_m\}_m$ possess a norm-convergent subsequence. Hence, the multiplier T_n of $L^1(0, 1)$ is not a compact operator on $L^1(0, 1)$, for any n.

Next, a routine calculation shows that $T_n^2 = T_{2n}$, for all n; hence, the operator T_n^2 is not compact, for any n. However, the product of two weakly compact operators on an L^1-space is compact [2, Cor. 13, p. 510]. Thus, the multiplier T_n is not weakly compact, for any n. ∎

3. *Multipliers of $L^1(\Pi\{a_n, b_n\})$*

Let S be the product semigroup $\prod_{n=1}^{N} I_n$, $N \geqq 2$. If $\{u_{kn}\}_k$ is the approximate identity for $L^1(I_n)$ constructed in the proof of Theorem 1 from a strictly decreasing sequence $\{c_{kn}\}_k$ of real numbers in (a_n, b_n) then $\{u_k = \bigotimes_{n=1}^{N} u_{kn}\}_k$ is an approximate identity bounded by one for $L^1(S)$ which consists of functions whose Gelfand transforms have compact support. This fact, together with the regularity of $L^1(S)$, implies that $\mathfrak{M}(L^1(S))$ is a CMA and that its canonical image in $M(\Gamma)$ is an L-subspace of $M(\Gamma)$ [12]. Moreover, $\hat{\bigotimes}_{n=1}^{N} \mathfrak{M}(L^1(I_n))$ is imbedded as an L-subalgebra in $\mathfrak{M}\left(\hat{\bigotimes}_{n=1}^{N} L^1(I_n)\right)=\mathfrak{M}(L^1(S))$ [8, Prop. 2.5.2].

Now, for each n, $1\leqq n\leqq N$, Γ_n denotes the structure semigroup of $L^1(I_n)$, and $\nu^{(n)}\colon L^1(I_n)\to M(\Gamma_n)$ denotes the canonical isometric imbedding of $L^1(I_n)$ in $M(\Gamma_n)$. Further, if $A_{\Gamma_n}=\{\nu_{f_n}^{(n)}\in M(\Gamma_n)\colon f_n\in L^1(I_n)\}$, then the map $(f_n)\mapsto(\nu_{f_n}^{(n)})$ defines an isometric isomorphism from $\prod_{n=1}^{N} L^1(I_n)$ onto $\prod_{n=1}^{N} A_{\Gamma_n}$. It is readily established that $\otimes\nu^{(n)}$ is an isometric isomorphism from the dense subspace $\bigotimes_{n=1}^{N} L^1(I_n)$ of $L^1(S)$ onto the dense subspace $\bigotimes_{n=1}^{N} A_{\Gamma_n}$ of $\hat{\bigotimes}_{n=1}^{N} A_{\Gamma_n}=A_\Gamma=\{\nu_f\in M(\Gamma)\colon f\in L^1(S)\}$, and that $\otimes\nu^{(n)}$ has a unique continuous extension to an isometric isomorphism $\nu=\hat{\otimes}\nu^{(n)}$ from $L^1(S)$ onto A_Γ.

The following notation will simplify the statement and proof of the upcoming theorem. For $1\leqq m\leqq N$, let $\Gamma'_m=\{t=(t_n)\in\Gamma\colon t_m=0_m\}$ and let $I'_m=\Pi_{n\neq m}I_n$. Further, for each m, let id_m denote the identity operator on $L^1(a_m, b_m)$. Finally, if $\mu\in M(\Gamma)$, then $\operatorname{supp}(\mu)$ will denote its support.

Theorem 3. *If $T\colon L^1(S)\to L^1(S)$ is a multiplier of $L^1(S)$ and μ_T is the measure in $M(\Gamma)$ corresponding to T, then μ_T admits the decomposition:*

$$\mu_T=\nu_f+\sum_{m=1}^{N}\tau_m,$$

where $f\in L^1(S)$ (i.e., $\nu_f\in A_\Gamma$) and, for $1\leqq m\leqq N$, τ_m is a measure in $M(\Gamma)$ with $\operatorname{supp}(\tau_m)\subseteq\Gamma'_m$ and corresponds to a multiplier T_m of $L^1(S)$ contained in $\mathbf{C}\cdot\mathrm{id}_m\otimes\mathfrak{M}(L^1(I'_m))$. Therefore, $\mathfrak{M}(L^1(S))=\hat{\bigotimes}_{n=1}^{N}\mathfrak{M}(L^1(a_n, b_n))$.

Proof. The last statement follows from the first by using induction on N. To prove the first statement of the theorem, fix T in $\mathfrak{M}(L^1(S))$ and let μ_T be the corresponding measure in $M(\Gamma)$. Let $F=\bigcup_{m=1}^{N}\Gamma'_m$ (the union of the faces) and define two new measures μ_T^+, μ_T^0 on Γ by setting $\mu_T^0=K(F)\mu_T$ and $\mu_T^+=\mu_T-\mu_T^0$. We will show that $\mu_T^+\in A_\Gamma$ (hence, $\mu_T^+=\nu_f$, for some f in $L^1(S)$), and that $\mu_T^0=\sum_{m=1}^{N}\tau_m$, with τ_m as in the statement of the theorem. We begin with μ_T^+.

For each k in $\mathbf{N}$, let $\omega_k=\nu_{Tu_k}\in A_\Gamma$. Now since $\nu_{Tf}=\mu_T*\nu_f$, for all f in $L^1(S)$, it follows that, for each ψ_x in $\hat{\Gamma}$, where $x=(x_n)\in\Pi(a_n, b_n]$,

$$\langle\psi_x,\mu_T\rangle\langle\psi_x,\nu_f\rangle=\langle\psi_x,\mu_T*\nu_f\rangle=\langle\psi_x,\nu_{Tf}\rangle; \tag{11}$$

that is

$$\mu_T(\Pi[0_n, x_n^+])\hat{f}(x) = \nu_{Tf}(\Pi[0, x_n^+]). \tag{12}$$

Setting $f = u_k = \otimes u_{kn}$ in equation (12) yields the identity:

$$\mu_T(\Pi[0_n, x_n^+]) = \omega_k(\Pi[0_n, x_n^+]), \tag{13}$$

valid for all (x_n) in $\Pi[c_{kn}, b_n]$. Using equation (13) and the fact that μ_T and ω_k are regular Borel measures on Γ, a straightforward but tedious computation shows that

$$\mu_T(\Pi[c_{kn}^+, x_n^+]) = \omega_k(\Pi[c_{kn}^+, x_n^+]), \tag{14}$$

for all (x_n) in $\Pi(c_{kn}, b_n]$. The computation is indicated for $N=2$ only; the general case is similar. For $N=2$, the set $\prod_{n=1}^{2}[0_n, x_n^+)$ may be written as a disjoint union of sets as follows:

$$[O_1, x_1^+)\times[O_2, x_2^+) = [O_1, c_{k1}^+)\times[O_2, c_{k2}^+)\cup \tag{15}$$

$$\cup[O_1, c_{k1}^+)\times[c_{k2}^+, x_2^+)\cup[c_{k1}^+, x_1^+)\times[O_2, c_{k2}^+)\cup[c_{k1}^+, x_1^+)\times[c_{k2}^+, x_2^+).$$

By (13), μ_T and ω_k agree on all of the sets in equation (15), except the final set; hence, they agree on the final set as well, establishing equation (14). For general N, the set $\prod_{n=1}^{N}[O_n, x_n^+)$ decomposes into 2^N disjoint sets, one of which is $\prod_{n=1}^{N}[c_{kn}^+, x_n^+)$. Since equation (13) implies that μ_T and ω_k agree on $\Pi[O_n, x_n^+)$ and on all of the other 2^N-1 sets in the decomposition of $\Pi[O_n, x_n^+)$, the identity (14) follows.

Next, for k in $\mathbf{N}$, set $R_k = \prod_{n=1}^{N}[c_{kn}^+, 1_n]\subseteq\Gamma$ and define measures $\mu_k = K(R_k)\mu_T$ and $\lambda_k = K(R_k)\omega_k$ on Γ. Then, for each $\psi_x\in\hat{\Gamma}$, $x=(x_n)\in\Pi(a_n, b_n]$,

$$\langle\psi_x, \mu_k\rangle = \int_\Gamma \psi_x\, d\mu_k = \int_\Gamma K(\Pi[O_n, x_n^+))K(R_k)\, d\mu_T = \tag{16}$$

$$= \begin{cases} 0, & \text{if} \quad x_n \leqq c_{kn}, \quad \text{for some } n \\ \mu_T(\Pi[c_{kn}^+, x_n^+)), & \text{if} \quad (x_n)\in\Pi(c_{kn}^+, b_n], \end{cases}$$

and, analogously,

$$\langle\psi_x, \lambda_k\rangle = \begin{cases} 0, & \text{if} \quad x_n \leqq c_{kn}, \quad \text{for some } n \\ \omega_k(\Pi[c_{kn}^+, x_n^+)), & \text{if} \quad (x_n)\in\Pi(c_{kn}^+, b_n]. \end{cases} \tag{17}$$

Therefore, by equation (14), μ_k and λ_k agree on $\hat{\Gamma}$ and, as a result $\mu_k=\lambda_k$, implying that $\mu_k\in A_\Gamma$, for every k.

Now, because A_Γ is a norm-closed subspace of $M(\Gamma)$, $\mu_T^+\in A_\Gamma$ provided the sequence $\{\mu_k\}_k$ of measures in A_Γ converges in norm to μ_T^+. To see that this occurs, first observe that $\mu_T^+ - \mu_k = \mu_T - K(F)\mu_T - K(R_k)\mu_T = K((\Gamma\setminus F)\setminus R_k)\mu_T$, so

$$\|\mu_T^+ - \mu_k\| = |\mu_T^+ - \mu_k|(\Gamma) = |\mu_T|((\Gamma\setminus F)\setminus R_k). \tag{18}$$

However, the product interval $\Gamma\setminus F = \prod_{n=1}^{N}(0_n, 1_n]$ is the countable disjoint union $\bigcup_{k=0}^{\infty}(R_{k+1}\setminus R_k)$, when R_0 is taken to be the empty set. Hence, because $|\mu_T|$ is a finite regular Borel measure on Γ,

$$\sum_{k=0}^{\infty}|\mu_T|(R_{k+1}\setminus R_k) = |\mu_T|(\Gamma\setminus F) < +\infty, \tag{19}$$

and so, for every $\varepsilon > 0$, there exists an integer K such that

$$|\mu_T|((\Gamma\setminus F)\setminus R_K) = \sum_{k=K}^{\infty}|\mu_T|(R_{k+1}\setminus R_k) < \varepsilon. \tag{20}$$

By equation (18), the inequality in (20) implies that $\|\mu_T^+ - \mu_k\| < \varepsilon$, for all $k > K$. Thus, $\mu_T^+ \in A_\Gamma$, as desired, and there exists an element f in $L^1(S)$ such that $\mu_T^+ = \nu_f$.

Turning now to $\mu_T^0 = K(F)\mu_T$, recall that

$$F = \bigcup_{m=1}^{N}\Gamma_m' = \Gamma_1' \cup (\Gamma_2'\setminus\Gamma_1') \cup \ldots \cup \left(\Gamma_m'\setminus\left(\bigcup_{j=1}^{m-1}\Gamma_j'\right)\right) \cup \ldots \cup \left(\Gamma_N'\setminus\left(\bigcup_{j=1}^{N-1}\Gamma_j'\right)\right).$$

Define measures τ_m, $1 \leqq m \leqq N$, in $M(\Gamma)$ as follows:

$$\tau_1 = K(\Gamma_1')\mu_T, \quad \tau_2 = K(\Gamma_2'\setminus\Gamma_1')\mu_T, \ \ldots, \ \tau_N = K\left(\Gamma_N'\setminus\left(\bigcup_{j=1}^{N-1}\Gamma_j'\right)\right)\mu_T.$$

Observe that $\mu_T^0 = \sum_{m=1}^{N}\tau_m$. Since $\mathfrak{M}(L^1(S))$ is a convolution measure algebra, each measure τ_m, $1 \leqq m \leqq N$, being absolutely continuous with respect to μ_T, corresponds to a multiplier of $L^1(S)$. It suffices therefore to determine the form of an arbitrary multiplier measure τ satisfying $\operatorname{supp}(\tau) \subseteq \Gamma_m'$, for some m, $1 \leqq m \leqq N$.

Because $L^1(S) = L^1(a_m, b_m)\hat{\otimes} L^1(I_m')$, without loss of generality, assume that $m = 1$ and that the support of the multiplier measure τ is contained in Γ_1'. It remains to prove that $\tau = \delta_1 \otimes \eta$, where δ_1 in $M(\Gamma_1)$ is the point mass at O_1, and η is a measure in $M\left(\prod_{n=2}^{N}\Gamma_n\right)$ corresponding to a multiplier of $L^1(I_1') = L^1\left(\prod_{n=2}^{N}I_n\right)$. Define η on the semicharacters $\psi_{x'} = \bigoplus_{n=2}^{N}\psi_{x_n}$, where $x' = (x_n)_{n=2}^{N} \in \prod_{n=2}^{N}(a_n, b_n]$, of $\prod_{n=2}^{N}\Gamma_n$ by the formula:

$$\langle\psi_{x'}, \eta\rangle = \langle\psi_{b_1}\otimes\psi_{x'}, \tau\rangle. \tag{21}$$

Since the linear span of such semicharacters $\psi_{x'}$ is dense in $C\left(\prod_{n=2}^{N}\Gamma_n\right)$, equation (21) defines a unique measure η in $M\left(\prod_{n=2}^{N}\Gamma_n\right)$. For each $x=(x_n)_{n=1}^{N}$ in $\prod_{n=1}^{N}(a_n, b_n)$,

$$\langle\psi_x, \delta_1\otimes\eta\rangle=\langle\psi_{x_1}, \delta_1\rangle\langle\psi_{x'}, \eta\rangle=\langle\psi_{x'}, \eta\rangle= \tag{22}$$

$$=\langle\psi_{b_1}\otimes\psi_{x'}, \tau\rangle=\langle\psi_x, \tau\rangle+\langle(\psi_{b_1}-\psi_{x_1})\otimes\psi_{x'}, \tau\rangle.$$

A routine calculation shows that $\operatorname{supp}((\psi_{b_1}-\psi_{x_1})\otimes\psi_{x'})\cap\operatorname{supp}(\tau)=\emptyset$; hence, equation (22) reduces to

$$\langle\psi_x, \delta_1\otimes\eta\rangle=\langle\psi_x, \tau\rangle, \quad \psi_x\in\hat{\Gamma}, \tag{23}$$

implying that $\tau=\delta_1\otimes\eta$. It remains to show that η corresponds to a multiplier of $L^1(I_1')$. To accomplish this, fix f in $L^1(a_1, b_1)$ with $\hat{f}(b_1)=\int_{a_1}^{b_1}f(t_1)\,dt_1=1$. Then, for each g in $L^1(I_1')$, $f\otimes g\in L^1(a_1, b_1)\otimes L^1(I_1')\subseteq L^1(S)$, and so $v_{f\otimes g}\in A_\Gamma$. If we let $v'=\hat{\bigotimes}_{n=2}^{N} v^{(n)}$ be the canonical imbedding of $L^1(I_1')=\hat{\bigotimes}_{n=2}^{N} L^1(a_n, b_n)$ in $\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}$, then, since $A_\Gamma=\hat{\bigotimes}_{n=1}^{N} A_{\Gamma_n}=A_{\Gamma_1}\hat{\otimes}\left(\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}\right)$, it follows that $v=v^{(1)}\hat{\otimes}v'$. This decomposition of v reveals that $v_{f\otimes g}=v_f^{(1)}\otimes v_g'\in A_{\Gamma_1}\otimes\left(\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}\right)$, for every g in $L^1(I_1')$. For each semi-character ψ_x, $x=(x_n)\in\prod_{n=1}^{N}(a_n, b_n]$, on Γ,

$$\langle\psi_x, \tau*v_{f\otimes g}\rangle=\langle\psi_{x_1}\otimes\psi_{x'}, \delta_1\otimes\eta\times\psi_{x_1}\otimes\psi_{x'}, v_f^{(1)}\otimes v_g'\rangle= \tag{24}$$

$$=\langle\psi_{x_1}, v_f^{(1)}\rangle(\langle\psi_{x'}, \eta\rangle\langle\psi_{x'}, v_g'\rangle)=\langle\psi_x, v_f^{(1)}\otimes(\eta*v_g')\rangle;$$

whence, it follows that

$$v_f^{(1)}\otimes(\eta*v_g')=\tau*v_{f\otimes g}, \tag{25}$$

for all g in $L^1(I_1')$. Because τ is a multiplier measure on A_Γ and $v_{f\otimes g}\in A_\Gamma$, equation (25) implies that $v_f^{(1)}\otimes(\eta*v_g')\in A_\Gamma$, for every g in $L^1(I_1')$.

Now, Θ_1: $A_{\Gamma_1}\hat{\otimes}\left(\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}\right)\to\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}$ defined by $\Theta_1\left(\sum_{i=1}^{\infty}\gamma_i\otimes\alpha_i\right)=\sum_{i=1}^{\infty}\langle\psi_{b_1}, \gamma_i\rangle\alpha_i$ is a continuous linear map from $A_{\Gamma_1}\hat{\otimes}\left(\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}\right)$ to $\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}$ [9, Lemma 2]. Therefore, for each $g\in L^1(I_1')$,

$$\Theta_1\big(v_f^{(1)}\otimes(\eta*v_g')\big)=\eta*v_g'\in\hat{\bigotimes}_{n=2}^{N} A_{\Gamma_n}. \tag{26}$$

Hence, η corresponds to a multiplier of $L^1(I_1')$. ∎

References

[1] G. AKST, L^1-algebras on semigroups, *Illinois J. Math.* **19** (1975), 454—466.
[2] N. DUNFORD and J. T. SCHWARTZ, Linear operators I, Pure and Applied Mathematics Series, vol. VII, *Interscience, New York*, 1958.
[3] J. GIL DE LAMADRID, Measures and tensors II, *Canad, Math. J.* **18** (1966), 762—793.
[4] A. GROTHENDIECK, Produits tensoriels topologiques et espaces nucléaires, *Amer. Math. Soc. Memoir, No. 16*, 1955.
[5] C. D. LAHR, Multipliers for certain convolution measure algebras, *Trans. Amer. Math. Soc.* **185** (1973), 165—181.
[6] L. J. LARDY, $L^1(a, b)$ with order convolution, *Studia Math.* **27** (1966), 1—8.
[7] R. LARSEN, The multipliers of $L^1([0, 1])$ with order convolution, *Publ. Math. (Debrecen)* **23** (1976), 239—248.
[8] J. L. TAYLOR, Measure algebras, *Regional Conference Series in Math.*, No. 16, *Amer. Math. Soc.*, 1973.
[9] J. TOMIYAMA, Tensor products of commutative Banach algebras, *Tôhoku Math. J.* **12** (1960), 147—154.
[10] J. G. WENDEL, Left centralizers and isomorphisms of group algebras, *Pacific J. Math.* **2** (1952), 251—261.
[11] K. YLINEN, Tensor products of complex L-spaces and convolution measure algebras, *Ann. Acad. Sci. Fenn. Series A* I. *Math., No. 558*, 1973.
[12] K. YLINEN, The multiplier algebra of a convolution measure algebra, *Pacific J. Math.* **51** (1974), 327—337.

(Received May 4, 1978.)