A generalized model of the Latin square design II.

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4. Introduction to the second part

In the first part of our paper [6] we gave a natural generalization of the usual model of the Latin square design in the third paragraph with the formulae (12) or (41). Then we proved theorems (1, 1′, 2, 3, 4, 5) valid in the new model. From these Theorem 1, Theorem 1′ and Theorem 2 are generalizations of theorems which are known in the usual model. Theorems 3, 4 and 5 (criteria) are in connection with the testing of statistical hypotheses. From these theorems one can see that the generalized assumption — the decomposition (12) — which corresponds to the initial condition of the usual Latin square design, is a fundamental requirement for each theorem.

For our generalized model the validity of the usual restrictions

$$\sum_{i=1}^{m} \lambda_i = \sum_{j=1}^{m} \nu_j = \sum_{h=1}^{m} \gamma_h = 0$$

was not generally assumed. One can find the meaning of the quantities λ_i, ν_j and γ_h in formula (1) in the first part of our paper. We took these assumptions into consideration only in the proofs of Theorem 1 and Theorem 1′. (The latter theorem can be obtained from Theorem 1.) The proof of Theorem 1 is based upon the formula (6), which may be proved by the earlier restrictions for the quantities λ_i, ν_j and γ_h.

We wish to remark that the validity of the usual restrictions for the sums of the quantities corresponding to the row-, the column — and the treatment — effects may be attained through the suitable transformations of the quantities implied in the expectations of random variables. Therefore we did not assume the validity of this relations in the first part of our paper.

The relations $\sum_{i=1}^{m} \lambda_i = \sum_{j=1}^{m} \nu_j = \sum_{h=1}^{m} \gamma_h = 0$ simplify the proofs of theorems 2, 3, 4 and 5 since on the basis of these we can realize for example the equalities $\mathbf{P}\boldsymbol{\lambda} = \mathbf{0}$, $\boldsymbol{\nu}^*\mathbf{P} = \mathbf{0}$ and $\mathbf{P}\boldsymbol{\Gamma} = \boldsymbol{\Gamma}\mathbf{P} = \mathbf{O}$. (The definition of $\mathbf{P}$ is given by formula (7), the definitions of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ and the matrix $\boldsymbol{\Gamma}$ can be found after (12). Here $\mathbf{0}$ denotes the m-dimensional zero column vector and $\mathbf{O}$ is the zero square matrix of order m.

Assuming the restrictions for the quantities λ_i, ν_j and γ_h the contents of theorems 3, 4 and 5 do not change, but their formulations will be given as follows.

Theorem 3″. *If the expectation of the matrix* ξ *exists and* $M(\xi)=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\boldsymbol{\Gamma}$, *further* $\mathbf{a}_0$ *is the righteigenvector of* $\mathbf{P}$ *which corresponds to the eigenvalue* 1, *then* $M(\boldsymbol{\eta}_2-\zeta)=\mathbf{O}$ *if and only if* $\boldsymbol{\lambda}=\mathbf{0}$. ($\boldsymbol{\eta}_2-\zeta$ is defined by (11).)

Theorem 4″. *If* $M(\xi)$ *exists and* $M(\xi)=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\boldsymbol{\Gamma}$, *where* $\mathbf{a}_0$ *is the right-eigenvector of the matrix* $\mathbf{P}$ *which belongs to its eigenvalue* 1, *so* $M(\boldsymbol{\eta}_1-\zeta)=\mathbf{O}$ (here $\boldsymbol{\eta}_1-\zeta$ — the matrix of discrepancies between columns — is given also by (11)) *if and only if* $\mathbf{v}=\mathbf{0}$.

Theorem 5″. *If the Latin square is a total symmetric, symmetric or cyclic one, moreover if* $M(\xi)$ *exists and has the decomposition* $M(\xi)=\mu\mathbf{a}_0\mathbf{a}_0^*+\boldsymbol{\lambda}\mathbf{a}_0^*+\mathbf{a}_0\mathbf{v}^*+\boldsymbol{\Gamma}$ *and* $\mathbf{P}\mathbf{a}_0=\mathbf{a}_0$, *then* $M(\boldsymbol{\eta}\ -\zeta)=\mathbf{O}$ *if and only if* $\boldsymbol{\Gamma}=\mathbf{O}$. (The meaning of $\boldsymbol{\eta}_h-\zeta$ is given by (11).)

According to Theorem 3″ the null hypothesis $H''_{\lambda 0}$ that the column vector of the row-effects is the zero vector (this means that row-effects do not exist) is equivalent to the null hypothesis $H'_{\lambda 0}$ according to which the expectation of the matrix of the discrepancies between the rows is the zero matrix.

On the basis of Theorem 4″, instead of the hypothesis H''_{v0} according to which the column vector of the column-effects is the zero vector, that is according to which there are no column-effects, we can take into consideration the hypothesis H'_{v0} which means that the expectation of the matrix of the discrepancies between columns equals the zero matrix.

Theorem 5″ contains the following equivalent hypotheses: $H''_{\Gamma 0}$ according to which the matrix of the treatment-effects is the zero matrix, that is the treatment-effects do not exist, the equivalent hypothesis $H'_{\Gamma 0}$ is the statement that the expectation of the matrix of the discrepancies between treatments equals the zero matrix.

In the second part of our paper we made an attempt at the reversion of Theorem 2. We wanted to prove that in the case of special Latin squares from the fact that the expectation of the random error matrix is the zero matrix, follows the decomposition (12) of the expectation matrix. We could not show the reverse of Theorem 2 and what is more, we gave a counter example in the case $m=3$ (m is the order of the square matrices and the Latin squares) of the cyclical Latin square design (see (23)). In this example we used, first of all, the method of minimum diadical representation of a matrix — this can be found in Egervary's paper [5] — for the determination of the general solution of the matrix equation (36). In the second part (Paragraph 5) the rearrangement of the forms of the equation (36) corresponding to the special Latin square designs — representation with the direct product — is a very essential step, namely we determine the general solution of the matrix equation (36) by the diadical decomposition of the rearranged coefficient matrix in which the direct product occurs.

In the second part of our paper we apply, on the one hand, the notations of the first part and, on the other hand, we introduce some additional notations.

$\otimes$ is the operational sign of the direct product. The direct product of the matrices $\mathbf{A}=\|a_{ij}\|_{i,j=\overline{1,m}}$ and $\mathbf{B}=\|b_{kl}\|_{k,l=\overline{1,u}}$ is defined with the equality

$$\mathbf{A}\otimes\mathbf{B}=\|\mathbf{A}b_{kl}\|_{k,l=1,n};$$

$\mathcal{A}, \mathcal{B}, \ldots$ hypermatrices.

Hypdet $\mathscr{A}$ is the hyperdeterminant of the hypermatrix $\mathscr{A}$. This is the determinant of the matrix, the elements of which are the blocks of the hypermatrix. (For example the hyperdeterminant of the hypermatrix $\mathscr{A}=\|\mathbf{A}_{ij}\|_{i,j=1,2,3}$, whose blocks are commutable in pairs, is

$$\operatorname{Hypdet}\mathscr{A} = \mathbf{A}_{11}\mathbf{A}_{22}\mathbf{A}_{33}+\mathbf{A}_{12}\mathbf{A}_{23}\mathbf{A}_{31}+\mathbf{A}_{13}\mathbf{A}_{21}\mathbf{A}_{32}-$$
$$-\mathbf{A}_{13}\mathbf{A}_{22}\mathbf{A}_{31}-\mathbf{A}_{11}\mathbf{A}_{23}\mathbf{A}_{32}-\mathbf{A}_{12}\mathbf{A}_{21}\mathbf{A}_{33},$$

where $\mathbf{A}_{ij}$ is a square matrix of order m.)

In the second part of our paper we continue the numbering of the formulae and the expressions with 61.

5. Decomposibility of $\mathbf{M}(\xi)$

We shall show — using certain results of Egervary's papers [3], [4] and [5] — that in our generalized model from the assumption that the expectation of the random error matrix is the zero matrix, the decomposibility of the form (12) does not follow even if $m=3$ and the Latin square is cyclic, namely it is given by (23).

Further, we shall first write down the forms (37), (38) and (39) of the matrix equation (36), which correspond to special (totally symmetric, cyclic and symmetric) Latin squares in terms of the direct products of the matrices using the following well-known theorem.

The matrix equation $\sum_{\mu}\sum_{\nu} c_{\mu\nu}\mathbf{A}^{\mu}\mathbf{X}\mathbf{B}^{\nu}=\mathbf{F}$, where $\mathbf{A}$ is a square matrix of order m, $\mathbf{B}$ is a square matrix of order n, $\mathbf{X}$ and $\mathbf{F}$ are matrices with m rows and n columns, the quantities $c_{\mu\nu}$ are constants, is equivalent to the system of equations with a direct polynomial coefficient

$$[\sum_{\nu}\sum_{\mu} c_{\mu\nu}\mathbf{A}^{\mu}\otimes(\mathbf{B}^{*})^{\nu}]\,\mathbf{x}=\mathbf{f}, \tag{61}$$

where $\mathbf{x}$ and $\mathbf{f}$ are mn-dimensional column vectors of the column vectors of the matrices $\mathbf{X}$ and $\mathbf{F}$ respectively, which will be written as hypervectors later.

Before the application of this theorem the matrix equations (37)—(39) will be rewritten in a form more favourable for us, multiplying each one by m and taking into consideration the equalities

$$(\mathbf{E}-\mathbf{P})M(\xi)(\mathbf{E}-\mathbf{P})+\mathbf{P}M(\xi)\mathbf{P} = M(\xi)-\mathbf{P}M(\xi)-M(\xi)\mathbf{P}+\mathbf{P}M(\xi)\mathbf{P},$$

$\mathbf{A}^m=\mathbf{\Omega}^m=\mathbf{E}$ and introducing the notation

$$\mathbf{K}[M(\xi)] = 2m\mathbf{P}M(\xi)\mathbf{P}-m\mathbf{P}M(\xi)-mM(\xi)\mathbf{P}+(m-1)M(\xi):$$

$$\mathbf{K}[M(\xi)]-\sum_{l=1}^{m-1}\mathbf{\Omega}^l M(\xi)\mathbf{\Omega}^l=\mathbf{O}, \tag{62}$$

$$\mathbf{K}[M(\xi)]-\sum_{l=1}^{m-1}\mathbf{\Omega}^l M(\xi)(\mathbf{\Omega}^{*})^l=\mathbf{O}, \tag{63}$$

$$\mathbf{K}[M(\xi)]-\sum_{l=1}^{m-1}\mathbf{A}^l M(\xi)\mathbf{A}^l=\mathbf{O}. \tag{64}$$

If the m-dimensional column vectors of $M(\xi)$ are $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_m$ and $\mathbf{0}$ is the m-dimensional zero column vector, then (62)—(64) can be written by the direct product on the basis of the formula (61) using the notation

$$\mathbf{K}(\otimes) = 2m\mathbf{P}\otimes\mathbf{P} - m\mathbf{P}\otimes\mathbf{E} - m\mathbf{E}\otimes\mathbf{P} + (m-1)\mathbf{E}\otimes\mathbf{E}:$$

(62′)
$$\left[\mathbf{K}(\otimes) - \sum_{l=1}^{m-1} \mathbf{\Omega}^l \otimes (\mathbf{\Omega}^*)^l\right] \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \vdots \\ \mathbf{m}_m \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix},$$

(63′)
$$\left[\mathbf{K}(\otimes) - \sum_{l=1}^{m-1} \mathbf{\Omega}^l \otimes \mathbf{\Omega}^l\right] \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \vdots \\ \mathbf{m}_m \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix},$$

(64′)
$$\left[\mathbf{K}(\otimes) - \sum_{l=1}^{m-1} \mathbf{A}^l \otimes (\mathbf{A}^*)^l\right] \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \vdots \\ \mathbf{m}_m \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}.$$

Now let $m=2$. Then the Latin squares (14) are totally symmetric (cyclic). In this case $\mathbf{\Omega}=\mathbf{\Omega}^*$ and the equation

(65)
$$[4\mathbf{P}\otimes\mathbf{P} - 2(\mathbf{P}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{P}) + \mathbf{E}\otimes\mathbf{E} - \mathbf{\Omega}\otimes\mathbf{\Omega}] \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

is given according to (62′) or (63′), where each of the matrices is a square matrix of order 2 and the vectors are 2-dimensional.

On the basis of the definition of the direct product for the coefficient matrix of the system of equations (65) we can easily get the hypermatrix

$$\begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix},$$

where $\mathbf{O}$ is the zero matrix of order 2. This means that (65) is satisfied by any kind of 4-dimensional hypervectors $\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}$. (The unknowns are the elements of $M(\xi)$.) From this it follows in the case of $m=2$ that the form (37) of (36) is satisfied by any kind of $M(\xi)$ not only by the matrix

$$M(\xi) = \mu\mathbf{a}_0\mathbf{a}_0^* + \lambda\mathbf{a}_0^* + \mathbf{a}_0 v^* + \Gamma.$$

(Here every vector is 2-dimensional and Γ is a square matrix of order 2.)

Consequently, in the case of $m=2$ the representation (12) of $M(\xi)$ does not follow from (36), that is the inverse statement of Theorem 2 is not true.

In the case of $m=3$ let the Latin square be cyclic, that is let us consider the Latin square (23). Then — from (63′) — the matrix equation (36) written by direct product is

$$(66)\qquad [6\mathbf{P}\otimes\mathbf{P}-3(\mathbf{P}\otimes\mathbf{E}+\mathbf{E}\otimes\mathbf{P})+2\mathbf{E}\otimes\mathbf{E}-\mathbf{\Omega}\otimes\mathbf{\Omega}-\mathbf{\Omega}^2\otimes\mathbf{\Omega}^2]\begin{pmatrix}\mathbf{m}_1\\ \mathbf{m}_2\\ \mathbf{m}_3\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix},$$

where the square matrices are of order 3 and the vectors are 3-dimensional. The coefficient matrix — denoted by $\mathscr{B}$ — of the system of equations (66) is a cyclic hypermatrix which consists of the square matrices $\mathbf{B}_1=\mathbf{E}-\mathbf{P}$, $\mathbf{B}_2=2\mathbf{P}-\mathbf{E}-\mathbf{\Omega}$ and $\mathbf{B}_3=2\mathbf{P}-\mathbf{E}-\mathbf{\Omega}^2$ of order 3, that is

$$\mathscr{B}=\begin{pmatrix}\mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3\\ \mathbf{B}_3 & \mathbf{B}_1 & \mathbf{B}_2\\ \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_1\end{pmatrix}.$$

Since $\mathbf{B}_2^*=\mathbf{B}_3$, therefore the cyclic hypermatrix $\mathscr{B}$ is symmetric, thus $\mathscr{B}^*=\mathscr{B}$. It is clear, that the non-trivial linearly independent solutions of the homogeneous system of linear equations (66) are the right-eigenvectors of $\mathscr{B}$ corresponding to its zero eigenvalue. So the general solution of (66) is a linear combination of the right-eigenvectors. The general solution of the original matrix equation, that is the general solution of (63) in the case $m=3$, can be obtained from the general solution (66) by the rearrangement of the hypervectors into square matrices. The fact that the coefficient matrix $\mathscr{B}$ of the system of equations (66) is symmetric makes the determination of the eigenvectors easier, since in this case we can take the following well-known theorems into consideration.

Theorem I. *The canonical representation of a real symmetric matrix* $\mathbf{A}$ *is*

$$\mathbf{A}=\sum_{k=1}^{s}\lambda_k(\mathbf{u}_{k1}\mathbf{u}_{k1}^*+\ldots+\mathbf{u}_{k\alpha_k}\mathbf{u}_{k\alpha_k}^*),$$

where λ_k *is that eigenvalue of* $\mathbf{A}$ *to which the right-eigenvectors* $\mathbf{u}_{k1},\ldots,\mathbf{u}_{k\alpha_k}$ *and the left-eigenvectors* $\mathbf{u}_{k1}^*,\ldots,\mathbf{u}_{k\alpha_k}^*$ *belong.*

Theorem II. *The minimal equation of a real symmetric matrix has only single real roots.*

The effective determination of the canonical form of a matrix $\mathbf{A}$ is performed in three steps.

1. First we calculate the eigenvalues (characteristic roots) $\lambda_1,\ldots,\lambda_s$ of $\mathbf{A}$ by solving the characteristic equation $\mathrm{Det}(\mathbf{E}\lambda-\mathbf{A})=0$. If also multiple characteristic roots were obtained then we should examine whether the minimal equation has only single roots.

2. In the second place we give the Lagrange polynomials $L_1(z),\ldots,L_s(z)$ corresponding to the single roots of the minimal equation (these are interpolating polynomials on the places $\lambda_1,\ldots,\lambda_s$) and in this way the Lagrange matrix polynomials $L_1(\mathbf{A}),\ldots,L_s(\mathbf{A})$ can be obtained. Then

$$\mathbf{A}=\sum_{k=1}^{s}\lambda_k L_k(\mathbf{A}).$$

3. As a third step we determine the eigenvectors corresponding to the eigenvalues λ_k ($k=\overline{1, s}$) with the method of diadical decomposition of the matrices $L_k(\mathbf{A})$. The diadical decomposition of $L_k(\mathbf{A})$ provides just as many eigenvectors as the multiplicity of λ_k.

Remark 7. The notion of a diad and the diadical representation (decomposition) occur in [5] (formulae (1), (2) on page 14).

Remark 8. We determine the eigenvalues of $\mathscr{B}$ on the basis of the theorem concerning the calculation of the determinant of a hypermatrix ([4], page 214).

According to Theorem II from the symmetry of $\mathscr{B}$ we get that the minimal equation has only single roots. Therefore no separate discussion of this problem is needed.

Remark 9. In [3] Egerváry generalized the canonical representation of a matrix for the case of a matrix function.

Remark 10. The method on the basis of which we give the matrix $L_k(\mathbf{A})$ with a minimal number of diads can be found in Egervary's paper [5] in detail. (See formulae (9.1) and (11).) The eigenvectors will be determined by this method.

1′. First calculate the eigenvectors of $\mathscr{B}$ from its characteristic equation $\operatorname{Det}(\mathscr{E}\lambda-\mathscr{B})=0$. It is easy to see that the square hypermatrix $\mathscr{E}\lambda-\mathscr{B}$ of order 9 is cyclically built from the matrices (blocks) of order 3 $\mathbf{A}_1=(\lambda-1)\mathbf{E}+\mathbf{P}$, $\mathbf{A}_2=\mathbf{E}+\mathbf{\Omega}-2\mathbf{P}$ and $\mathbf{A}_3=\mathbf{E}+\mathbf{\Omega}^2-2\mathbf{P}$, where $\mathbf{E}$ is the identity matrix, $\mathbf{\Omega}$ is the primitive cyclic matrix of order 3, $\mathbf{P}$ is given by (13) and $\mathscr{E}$ is the identity matrix of order 9, that is

$$\mathscr{E}\lambda-\mathscr{B}=\begin{pmatrix}\mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3\\ \mathbf{A}_3 & \mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_1\end{pmatrix}.$$

Since $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are also cyclic matrices, $\mathscr{E}\lambda-\mathscr{B}$ consists of blocks which are commutable in pairs. The determinant of such a matrix can be calculated on the basis of the formulae

$$\operatorname{Det}(\mathscr{E}\lambda-\mathscr{B})=\operatorname{Det}[\operatorname{Hypdet}(\mathscr{E}\lambda-\mathscr{B})]$$

and

$$\operatorname{Hipdet}(\mathscr{E}\lambda-\mathscr{B})=\mathbf{A}_1^3+\mathbf{A}_2^3+\mathbf{A}_3^3-3\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3.$$

It is easily seen, that $\mathbf{A}_1^3=(\lambda^3-3\lambda^2+3\lambda-1)\mathbf{E}+(3\lambda^2-3\lambda+1)\mathbf{P}$, or

$$\mathbf{A}_1^3=\begin{pmatrix}a_1 & a_2 & a_2\\ a_2 & a_1 & a_2\\ a_2 & a_2 & a_1\end{pmatrix}$$

with the notations $a_1=\lambda^3-2\lambda^2+2\lambda-\dfrac{2}{3}$ and $a_2=\lambda^2-\lambda+\dfrac{1}{3}$;

$$\mathbf{A}_2^3=2\mathbf{E}+3\mathbf{\Omega}+3\mathbf{\Omega}^2-8\mathbf{P},$$

that is

$$A_2^3 = \frac{1}{3}\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix};$$

$A_3^3 = A_2^3$;

$$-3A_1A_2A_3 = -3(\lambda-1)(2E-4P+\Omega+\Omega^2),$$

that is

$$-3A_1A_2A_3 = \begin{pmatrix} b_1 & b_2 & b_2 \\ b_2 & b_1 & b_2 \\ b_2 & b_2 & b_1 \end{pmatrix},$$

and here $b_1=2-2\lambda$, $b_2=\lambda-1$. Using these formulae

$$\text{Hipdet}\,(\mathcal{E}\lambda-\mathcal{B}) = \begin{pmatrix} c_1 & c_2 & c_2 \\ c_2 & c_1 & c_2 \\ c_2 & c_2 & c_1 \end{pmatrix},$$

where $c_1=\lambda^3-2\lambda^2$, $c_2=\lambda^2$. Finally

$$\text{Det}\,(\mathcal{E}\lambda-\mathcal{B}) = \lambda^7(\lambda^2-6\lambda+9).$$

From this it follows that $\lambda_1=0$ is a septuple eigenvalue of $\mathcal{B}$ while $\lambda_2=3$ is a double eigenvalue. The minimal equation of $\mathcal{B}$ has only single roots, since $\mathcal{B}$ is symmetrical. (Theorem II.)

2'. From the minimal polynomial $\Delta(\lambda)=(\lambda-3)\lambda$ the Lagrange polynomial $L_1(z)$ corresponding to the eigenvalue $\lambda_1=0$ is

$$L_1(z) = 1-\frac{z}{3},$$

and the Lagrange matrix polynomial belonging to the eigenvalue $\lambda_1=0$ is

$$L_1(\mathcal{B}) = \mathcal{E}-\frac{1}{3}\mathcal{B},$$

whereas the Lagrange matrix polynomial belonging to the eigenvalue $\lambda_2=3$ is

$$L_2(\mathcal{B}) = \frac{1}{3}\mathcal{B}.$$

Finally the matrix $\mathcal{B}$ can be written by the Lagrange matrix polynomials in the form

$$\mathcal{B} = 0\cdot\left(\mathcal{E}-\frac{1}{3}\mathcal{B}\right)+3\cdot\frac{1}{3}\mathcal{B}. \tag{67}$$

3'. We must still determine the eigenvectors corresponding to the eigenvalues 0 and 3 to give the canonical representation of $\mathcal{B}$. Though the nontrivial solutions of the system of equations (66) are only the eigenvectors belonging to the eigenvalue 0, nevertheless we calculated the eigenvectors corresponding to the eigenvalue 3 for completeness. (The latter eigenvectors are no solutions of (66).) The eigenvectors are obtained by the diadical decompositions of the matrix polynomials $L_1(\mathscr{B})$ and $L_2(\mathscr{B})$ ([5], formulae (9.1) and (11).)

The diadical decomposition of $L_1(\mathscr{B})$ is:

$$L_1(\mathscr{B}) = \sum_{p=1}^{7} \mathbf{u}_{1p}\mathbf{u}_{1p}^*,$$

where

$$(68) \quad \mathbf{u}_{11} = \frac{\sqrt{7}}{21}\begin{pmatrix} 7\\1\\1\\1\\1\\-2\\1\\-2\\1 \end{pmatrix}, \quad \mathbf{u}_{12} = \frac{\sqrt{21}}{84}\begin{pmatrix} 0\\16\\2\\-5\\2\\3\\2\\3\\-5 \end{pmatrix}, \quad \mathbf{u}_{13} = \frac{\sqrt{3}}{12}\begin{pmatrix} 0\\0\\6\\1\\-2\\1\\-2\\1\\1 \end{pmatrix}, \quad \mathbf{u}_{14} = \frac{\sqrt{6}}{12}\begin{pmatrix} 0\\0\\0\\4\\1\\1\\1\\1\\-2 \end{pmatrix},$$

$$\mathbf{u}_{15} = \frac{\sqrt{10}}{20}\begin{pmatrix} 0\\0\\0\\0\\5\\1\\-3\\1\\2 \end{pmatrix}, \quad \mathbf{u}_{16} = \frac{\sqrt{15}}{15}\begin{pmatrix} 0\\0\\0\\0\\0\\3\\1\\-2\\1 \end{pmatrix}, \quad \mathbf{u}_{17} = \frac{\sqrt{3}}{3}\begin{pmatrix} 0\\0\\0\\0\\0\\0\\1\\1\\1 \end{pmatrix}.$$

On the other hand, the diadical decomposition of $L_2(\mathscr{B})$ — in accordance with the fact that $\lambda_2=3$ is a double eigenvalue — is $L_2(\mathscr{B})=\mathbf{u}_{21}\mathbf{u}_{21}^*+\mathbf{u}_{22}\mathbf{u}_{22}^*$,

$$(69) \quad \mathbf{u}_{21} = \frac{\sqrt{2}}{6}\begin{pmatrix} 2\\-1\\-1\\-1\\-1\\2\\-1\\2\\-1 \end{pmatrix}, \quad \mathbf{u}_{22} = \frac{\sqrt{6}}{6}\begin{pmatrix} 0\\1\\-1\\1\\-1\\0\\-1\\0\\1 \end{pmatrix}.$$

If we substitute these decompositions into (67), then the canonical representation of $\mathscr{B}$ is

$$\mathscr{B} = 0\cdot\sum_{p=1}^{7}\mathbf{u}_{1p}\mathbf{u}_{1p}^* + 3\cdot\sum_{q=1}^{2}\mathbf{u}_{2q}\mathbf{u}_{2q}^*,$$

where the eigenvectors are given by (68) and (69).

From these it follows that the general solution of (66) is

$$\mathbf{m} = \sum_{p=1}^{7} \varrho_p \mathbf{u}_{1p}, \tag{70}$$

where the quantities ϱ_p $(p=\overline{1,7})$ are constants, $\mathbf{u}_{1p}$ $(p=\overline{1,7})$ are given by (68), $\mathbf{m}$ is a 9-dimensional column vector, which is composed of vectors $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ (these are 3-dimensional column vectors of $M(\xi)$).

If we rewrite the equation (66) into the form (63) and consider the case $m=3$, then we get the equation

$$3(\mathbf{E}-\mathbf{P})M(\xi)(\mathbf{E}-\mathbf{P})+3\mathbf{P}M(\xi)\mathbf{P}+M(\xi)+ \\ +\mathbf{\Omega}M(\xi)\mathbf{\Omega}^2+\mathbf{\Omega}^2M(\xi)\mathbf{\Omega}=\mathbf{O} \tag{71}$$

in which each of the matrices is a square one of order 3.

From the general solution (70) of the equation (66) after rearrangement the general solution of (71) is

$$M(\xi) = \sum_{p=1}^{7} \delta_p \mathbf{M}_p, \tag{72}$$

the quantities δ_p are constants and the matrices $\mathbf{M}_p$ $(p=\overline{1,7})$ from (68) are

$$\mathbf{M}_1=\begin{pmatrix}7&1&1\\1&1&-2\\1&-2&1\end{pmatrix},\quad \mathbf{M}_2=\begin{pmatrix}0&-5&2\\16&2&3\\2&3&-5\end{pmatrix},\quad \mathbf{M}_3=\begin{pmatrix}0&1&-2\\0&-2&1\\6&1&1\end{pmatrix},\quad \mathbf{M}_4=\begin{pmatrix}0&4&1\\0&1&1\\0&1&-2\end{pmatrix},$$

$$\mathbf{M}_5=\begin{pmatrix}0&0&-3\\0&5&1\\0&1&2\end{pmatrix},\quad \mathbf{M}_6=\begin{pmatrix}0&0&1\\0&0&-2\\0&3&1\end{pmatrix},\quad \mathbf{M}_7=\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix}.$$

Remark 11. The matrices $\mathbf{M}_p$ $(p=\overline{1,7})$ are indeed solutions of the equation (71) which can be ascertained by substitutions.

The general solution of matrix equation (71) differs from the expected solution

$$M(\xi) = \mu\mathbf{a}_0\mathbf{a}_0^* + \lambda\mathbf{a}_0^* + \mathbf{a}_0\mathbf{v}^* + \mathbf{\Gamma},$$

for which it has been proved in Theorem 2 ([6]) that it is a solution of the equation (36). This means that in the case $m=3$ with a cyclical Latin square the decomposition (12) of $M(\xi)$ does not follow from that form (38) of the equation (36), which belongs to the cyclical case, that is the decomposition (72) can be obtained from (38). Consequently the inverse statement of theorem 2 ([6]) is not true even if $m=3$ and the Latin square design is cyclic, though our generalized model (see [6] Paragraph 3) — on the basis of the theorems proved in Paragraph 3 — seems to be a natural generalization of the usual model.

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(Received 29. December, 1979)