Theorem 3. *In a non-flat $HR—F_n$ which is also a $WR-F_n$ its recurrence vector λ_m is necessarily independent of $\dot{x}^i$'s. Conversely, a non-flat $HR-F_n$ with recurrence vector being independent of $\dot{x}^i$'s becomes a $WR-F_n$ provided it also admits* (2.8).

Now we consider a manifold F_n admitting (2.3). Analogous to the terminology adopted by Misra [4], a manifold F_n admitting (2.3) will be called a W^i_{jk}-recurrent manifold. Similarly we may define a W^i_j-recurrent manifold admitting (2.4).

Differentiating (2.3) partially with respect to $\dot{x}^h$ and using (1.12) we get

$$\dot{\partial}_h \mathscr{B}_m W^i_{jk} = (\dot{\partial}_h \lambda_m) W^i_{jk} + \lambda_m W^i_{jkh}. \tag{2.9}$$

Interchange of the operators of partial and covariant differentiation by means of the commutation formula exhibited by (1.5) for the tensor-field W^i_{jk}, and using (1.12), the equation (2.9) reduces to

$$\mathscr{B}_m W^i_{jkh} + W^r_{jk} G^i_{rhm} - W^i_{rk} G^r_{jhm} - W^i_{jr} G^r_{khm} = (\dot{\partial}_h \lambda_m) W^i_{jk} + \lambda_m W^i_{jkh}. \tag{2.10}$$

In view of (1.15) and (2.10) it is clear that a W^i_{jk}-recurrent Finsler manifold will be a $WR-F_n$ if and only if there holds the identity

$$W^r_{jk} G^i_{rhm} = W^i_{rk} G^r_{jhm} + W^i_{jr} G^r_{khm} + (\dot{\partial}_h \lambda_m) W^i_{jk}. \tag{2.11}$$

In view of Theorem 2 and the above conclusion we have the

Theorem 4. *For a $HR-F_n$ to be a $WR-F_n$* (2.11) *is a necessary and sufficient condition.*

Transvecting (2.11) with $\dot{x}^m$, and using (1.4) we have

$$(\dot{x}^m \dot{\partial}_h \lambda_m) W^i_{jk} = 0.$$

Thus, for a F_n admitting $W^i_{jk} \neq 0$ and (2.11), we have

$$\dot{x}^m \dot{\partial}_h \lambda_m = 0.$$

Putting $\lambda = \lambda_m \dot{x}^m$, the above relation reduces to

$$\lambda_m = \dot{\partial}_m \lambda. \tag{2.12}$$

This result gives rise to the

Theorem 5. *If a W^i_{jk}-recurrent manifold F_n becomes a $WR-F_n$ its recurrence vector λ_m satisfies* (2.12).

Next, let us consider a W^i_j-recurrent manifold characterized by (2.4). Differentiating (2.4) partially with respect to $\dot{x}^k$ we have

$$\dot{\partial}_k \mathscr{B}_m W^i_j = (\dot{\partial}_k \lambda_m) W^i_j + \lambda_m \dot{\partial}_k W^i_j. \tag{2.13}$$

Applying (1.5) for W^i_j this equation reduces to

$$\mathscr{B}_m \dot{\partial}_k W^i_j + W^r_j G^i_{rkm} - W^i_r G^r_{jkm} = (\dot{\partial}_k \lambda_m) W^i_j + \lambda_m \dot{\partial}_k W^i_j.$$

Taking skew-symmetric part in k, j, multiplying by 1/3 throughout the above equation, and using the symmetric property of G^i_{jkh} in its lower indices and (1.12), we get

$$\mathscr{B}_m W^i_{jk} + \frac{2}{3} W^r_{[j} G^i_{k]rm} = \frac{2}{3} W^i_{[j} \dot\partial_{k]} \lambda_m + \lambda_m W^i_{jk}.$$

Thus, for a W^i_j-recurrent Finsler manifold to be a W^i_{jk}-recurrent Finsler manifold the existence of relation

$$W^r_{[j} G^i_{k]rm} = W^i_{[j} \dot\partial_{k]} \lambda_m \tag{2.14}$$

is a necessary and sufficient condition. Therefore, we have the

Theorem 6. *A W^i_j-recurrent Finsler manifold becomes W^i_{jk}-recurrent if and only if it admits* (2.14).

§ 3. Possibilities for a $WR-F_n$ to be a $HR-F_n$. In this section we discuss the possibilities for a projective recurrent Finsler manifold F_n to be a recurrent Finsler manifold. Let us consider a projective recurrent Finsler manifold F_n admitting (1.15). Differentiating (1.9) covariantly with respect to x^m, and using (1.15) we have

$$\lambda_m W^i_{jkh} = \mathscr{B}_m H^i_{jkh} - \frac{1}{n+1} \{\delta^i_h \mathscr{B}_m H^r_{jkr} + \dot x^i \mathscr{B}_m \dot\partial_h H^r_{jkr}\} + \\ + \frac{2}{n^2-1} \{n \mathscr{B}_m \dot\partial_h H_{[j} + \mathscr{B}_m H_{h[j} + \dot x^r \mathscr{B}_m \dot\partial_h H_{r[j}\} \delta^i_{k]}.$$

Applying the commutation formula (1.5), and (1.4) the above identity reduces to

$$\lambda_m W^i_{jkh} = \mathscr{B}_m H^i_{jkh} - \frac{1}{n+1} \{\delta^i_h \mathscr{B}_m H^r_{jkr} + \dot x^i (\dot\partial_h \mathscr{B}_m H^r_{jkr} + H^r_{skr} G^s_{hmj} + H^r_{jsr} G^s_{hmk})\} + \\ + \frac{2}{n^2-1} \{n \mathscr{B}_m \dot\partial_h H_{[j} + \mathscr{B}_m H_{h[j} + \dot x^r (\dot\partial_h \mathscr{B}_m H_{r[j} + H_{rs} G^s_{hm[j})\} \delta^i_{k]}. \tag{3.1}$$

However, if the manifold under consideration also admits (1.14c) so that the tensor-fields H^r_{jkr}, $\dot\partial_h H_j$ are also recurrent*), the above identity simplifies to

$$\lambda_m H^i_{jkh} = \mathscr{B}_m H^i_{jkh} - \frac{1}{n+1} \dot x^i \{H^r_{jkr} (\dot\partial_h \lambda_m) + H^r_{skr} G^s_{hmj} + H^r_{jsr} G^s_{hmk}\} + \\ + \frac{2}{n^2-1} \dot x^r \{(\dot\partial_h \lambda_m) H_{r[j} + H_{rs} G^s_{hm[j}\} \delta^i_{k]}, \tag{3.2}$$

where (1.9) is used. In view of (1.9) and (3.2) it is clear that the projective recurrent manifold F_n satisfying (1.14c) will be recurrent if and only if there holds the identity

$$\dot x^i \{H^r_{jkr} (\dot\partial_h \lambda_m) + H^r_{skr} G^s_{hmj} + H^r_{jsr} G^s_{hmk}\} - \\ - \frac{2}{n-1} \dot x^r \{(\dot\partial_h \lambda_m) H_{r[j} + H_{rs} G^s_{hm[j}\} \delta^i_{k]} = 0. \tag{3.3}$$

*) By definition, $\dot\partial_h H_j = H_{jh}$ [7].

Conversely, if a $WR-F_n$ is $HR-F_n$ so that there holds both the equations (1.15) and (1.13) it is then easily seen from (1.13) that it also satisfies (1.14c). Consequently (3.3) results from (3.2). Thus, we conclude the

Theorem 7. *A projective recurrent Finsler manifold F_n is recurrent if and only if there holds the identities* (1.14c) *and* (3.3).

Transvecting (3.2) with $\dot{x}^h$, using (1.4), (1.7a), and homogeneous property of λ_m we have

$$\mathscr{B}_m H^i_{jk} = \lambda_m H^i_{jk}. \tag{3.4}$$

Thus, a $WR-F_n$ satisfying (1.14c) is necessarily a H^i_{jk}-recurrent manifold. Conversely, when $WR-F_n$ is H^i_{jk}-recurrent there follows from (3.1) under transvection with $\dot{x}^h$ and making use of (1.2), (1.4), (1.7a), (1.10), and homogeneous properties of Berwald's curvature tensor and its associate tensors:

$$-\dot{x}^i(\mathscr{B}_m-\lambda_m)H_{jkr}+\frac{2}{n-1}\,\dot{x}^h(\mathscr{B}_m-\lambda_m)H_{h[j}\delta^i_{k]}=0.$$

Multiplying it by $\dot{x}^k$, applying (1.8b), (1.8c) and the fact that H_j is also recurrent under the hypothesis we get

$$(\mathscr{B}_m-\lambda_m)\dot{x}^k H_{kj}=0, \tag{3.5}$$

which proves that $\dot{x}^k H_{kj}$ is recurrent under the hypothesis and not H_{kj}. Thus, we have the

Theorem 8. *The condition expressed by* (1.14c) *is sufficient to reduce a $WR-F_n$ into a H^i_{jk}-recurrent manifold but not necessary.*

The second author [4] proved that a H^i_{jk}-recurrent Finsler manifold becomes a recurrent Finsler manifold if and only if

$$H^i_{jk}\dot{\partial}_h\lambda_m = H^r_{jk}G^i_{rhm}+2H^i_{r[j}G^r_{k]hm}. \tag{3.6}$$

In view of this result we may conclude the

Theorem 9. *A $WR-F_n$ admitting* (1.14c) *is a $HR-F_n$ if and only if it admits* (3.6).

Transvecting (3.5) with $\dot{x}^k$, and using (1.7b) we have

$$\mathscr{B}_m H^i_j = \lambda_m H^i_j.$$

Thus, we conclude that in a projective recurrent manifold admitting (1.14c) the tensor H^i_j is also recurrent. Using the theorem [4]: An H^i_j-recurrent F_n admitting vanishing covariant derivative of the tensor-field G^i_{jkh} is an H-recurrent space, we conclude the

Theorem 10. *A projective recurrent Finsler manifold admitting* (1.14c) *and vanishing covariant derivative of the tensor-field G^i_{jkh} is a recurrent manifold.*

§ 4. Associate projective curvature tensor W_{jkhm}. Let us consider a manifold F_n admitting (1.15). Transvecting (1.15) with g_{ip} and using

$$W^i_{jkh} g_{ip} = W_{jkhp}, \tag{4.1}$$

we have

$$\mathscr{B}_m W_{jkhp} - (\mathscr{B}_m g_{ip}) W^i_{jkh} = \lambda_m W_{jkhp}. \tag{4.2}$$

Thus, the associate projective curvature tensor W_{jkhp} is not, in general, recurrent in a $WR-F_n$. However, if the manifold is affinely connected so that there holds [7, pp. 80—81]

$$\mathscr{B}_m g_{ip} = 0, \tag{4.3}$$

it follows from (4.2) that

$$\mathscr{B}_m W_{jkhp} = \lambda_m W_{jkhp}.$$

Thus, we have the

Theorem 11. *In an affinely connected $WR-F_n$ the associate projective curvature tensor W_{jkhp} given by* (4.1) *is recurrent.*

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Appendix

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ALLAHABAD,
ALLAHABAD (INDIA).

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