On the monotone classes of maximal limit-logic M

By ÉVA GÁRDOS (Debrecen)

1. Introduction

The notion of limit-logic was introduced by S. V. YABLONSKI [13] in 1958. The necessity of this notion arose in the study of finite-valued logics [12] and infinite-valued logics [14]. In particular, the infinite-valued logics contain as many as continuum functions, thus, working with these logics is very difficult. Therefore the necessity of a logic was recognized which contains countably many functions and can be regarded, from a certain point of view, as the model of all k-valued logics. Contrary to the many- and the infinite-valued logics, the model of limit-logics has many realizations, or rather there are continuum, pairwise non-isomorphic limit-logics [1]. In the first place the equivalent classes are examined, and, at the same time, the question arises with respect to every partial ordering, whether or not there is a maximal and a minimal element. The best known and simplest equivalence is isomorphism. From the references (6, 8, 13) it is well-known that there are as many as continuum, pairwise non-isomorphic limit-logics. It has been proved [1] that no maximal and minimal elements exist under the well-known algebraic ways of ordering. The partial ordering introduced in [1, 11] in a logical way also decomposes the set of limit-logics into equivalence classes, and under such a partial ordering there are already maximum and minimum limit-logics. Such a logic is maximal to the extent that it contains every other limit-logic, and is minimal if it is contained in every limit-logic. S. V. Yablonski [12] examined the pre-complete classes of the k-valued logic. These classes have been investigated by him and many others [4, 5, 7, 14, 18].

The aim of the paper is to examine certain classes studied by Yablonski in a maximal limit-logic M with help of the method given in [12]. In the k-valued logic there are finite pre-complete classes. In our paper we shall prove that in the limit-logic M there are continuum monotone pre-complete classes. We suceeded in generalizing Yablonski's concept of monotone pre-complete classes and we have reached the following results:

a) in a limit-logic M the monotone function class belonging to a linear order r is pre-complete.

b) a limit-logic M contains continuum pre-complete classes. Moreover, the number of the pre-complete monotone function classes with respect to linear order is continuum.

2. Elements

Definition 2.1 [4]. Let E_k be an arbitrary set of k elements. Denote by P_k^n $(n=0, 1, 2, \ldots)$ the set of functions with n variables, with the variables and values taken from the set E_k. The function set $P_k=\bigcup_{n=0}^{\infty} P_k^n$ will be called d-valued logic. Without loss of generality we suppose $E_k=\{0, 1, \ldots, k-1\}$, $k\geqq 2$.

Definition 2.2 If the finite set E_k in Definition 2.1 is replaced by the countably infinite set, $E_{\aleph_0}$, the function-set $P_{\aleph_0}$ will be called the infinite-valued logic. In this paper, $E_{\aleph_0}$ will always be the non-negative integers, that is

$$E_{\aleph_0} = \{0, 1, 2, \ldots, n, n+1, \ldots\}.$$

Definition 2.3 [6]. The subset P of the infinite-valued logic $(P_{\aleph_0})$ will be called limit-logic, if

a) in the function-set P there are countably many functions,

b) for every natural number k $(k\geqq 2)$ there exists a function-set A_k $(A_k\subseteqq P)$, which can be mapped onto the k-valued logic. Let P be a limit-logic, ε a subset of $E_{\aleph_0}(\varepsilon\subseteqq E_{\aleph_0})$ and $g(x_1, \ldots, x_n)\in P$.

Definition 2.4. Put

$$g_\varepsilon(x_1, \ldots, x_n) = \begin{cases} g(x_1, \ldots, x_n) & \text{if} \quad x_1, \ldots, x_n\in\varepsilon^n(\varepsilon x \ldots x\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

The function $g_\varepsilon(x_1, \ldots, x_n)$ will be called the restriction of the function $g(x_1, \ldots, x_n)$ to ε.

Definition 2.5. We say that the function set A_k is the model of the k-valued logic on the set $\varepsilon_k=\{e_0, e_1, \ldots, e_{k-1}\}$ $(k\geqq 2)$, $(\varepsilon_k\subset E_{\aleph_0})$, if in the set $[A_k]$ there is such a function $f(x_1, x_2)$ that

$$f_{\varepsilon_k}(x_1, x_2) = \begin{cases} e_{i+1} & \text{if} \quad (x_1, x_2)\in\varepsilon_k\times\varepsilon_k \quad \text{and} \quad \max(x_1, x_2) = e_i, \\ & \text{where} \quad 0\leqq i\leqq k-2; \\ e_0, & \text{if} \quad (x_1, x_2)\in\varepsilon_k\times\varepsilon_k \quad \text{and} \quad \max(x_1, x_2) = e_{k-1}. \end{cases}$$

It is easy to see that if A_k is the model of the k-valued logic on the set ε_k, then in the function set A_k there is at least one function-set, whose restriction to the set ε_k is isomorphic with the k-valued logic.

Definition 2.6 [2]. The system of the finite subsets ε_k will be called the domain of the function-set A and we denote it by T_A. The set $\varepsilon_k=\{e_0, e_1, \ldots, e_{k-1}\}$ belongs to the domain T_A iff it is model of the k-valued logic on the set ε_k.

Definition 2.7 [2]. The limit-logic P is called increasing, if its domain T_P contains an increasing sequence of the finite sets

$$(\Pi = \{\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots)\} \quad (\varepsilon_i\subset\varepsilon_{i+1} \ (i=2, 3, \ldots)).$$

Theorem 2.1 [2]. *The limit-logic is maximal if and only if it is increasing.*

The notions of superposition, closure and pre-completeness are defined as in [12].

3. § The limit-logic M

We shall define a representation M of the maximal limit logics which we are going to examine in detail:

Define the function $\mu_k(x_1, x_2) \in P_{\aleph_0} (k \geqq 2)$ as follows:

$$\mu_k(x_1, x_2) = \begin{cases} e, & \text{if } (x_1, x_2) \in \varepsilon_k \times \varepsilon_k \text{ and } \max(x_1, x_2) = e-1, \\ & \text{where } 1 \leqq e-1 \leqq k-1; \\ 1, & \text{if } (x_1, x_2) \in \varepsilon_k \times \varepsilon_k \text{ and } \max(x_1, x_2) = k \\ 0, & \text{otherwise.} \end{cases}$$

Let M_k denote the closure of the function-set

$$\{\mu_k(x_1, x_2)\} \left([\{\mu_k \{x_1, x_2)\}] \right) \quad \text{and} \quad M = \left[\bigcup_{k=2}^{\infty} M_k \right].$$

Remark 3.1. It is easy to see that M_k is isomorphic with P_k and M is a limit-logic.

Remark 3.2. As an illustration we shall give the functions $\mu_2(x_1, x_2)$, $\mu_3(x_1, x_2)$ and $\mu_k(x_1, x_2)$.

$x_2 \backslash x_1$	0	1	2	3	4	…
0	0	0	0	0	0	…
1	0	2	1	0	0	…
2	0	1	1	⋮	⋮	
3	0	…	…	…	…	…
4	0	…	0	…	0	…
⋮	⋮		⋮		⋮	

$\mu_2(x_1, x_2)$ $\qquad \varepsilon_k = \{1, 2\}$

$x_2 \backslash x_1$	0	1	2	3	4	…
0	0	0	0	0	0	0 …
1	0	2	3	1		⋮
2	0	3	3	1		0 …
3	0	1	1	1		⋮
4	0	…	…	…	…	…
⋮	0	…	0	…		0 …
	⋮		⋮			⋮

$\mu_3(x_1, x_2)$ $\qquad \varepsilon_k = \{1, 2, 3\}$

$x_2 \backslash x_1$	0	1	2	3	4	…	$k-1$	k	$k+1$	…
0	0	0	0	0	0	…	0	0	0	…
1	0	2	3	4	5	…	k	1	0	…
2	0	3	3	4	5	…	k	1	0	…
3	0	4	4	4	5	…	k	1	0	…
4	0	5	5	5	5	…	k	1	0	…
·	·	………							·	…
·	·	………							·	…
·	·	………							·	…
$k-1$	0	k	k	k	k	…	k	1	0	…
k	0	1	1	1	1	…	1	1	0	…
$k+1$	0	………								
⋮	0		…	…	0	………			0	…
⋮	⋮				⋮				⋮	

$\mu_k(x_1, x_2)$

$\varepsilon_k = \{1, 2, \dots, k\}$

Theorem 3.1. *M is a limit-logic.*

Proof. By Definition 2.3 we have to show:

a) the function set M is countable, since $M = \left[\bigcup_{k=2}^{\infty} M_k\right]$, a countable union of countable sets,

b) for every natural number k ($k \geqq 2$) there is a function subset in M which can be mapped onto the k-valued logic.

This statement follows from the fact that the functions $\mu_k(x_1, x_2)$ can be brought into one to one correspondance with the $W_k(x_1, x_2)$ Webb functions [5]. Furthermore, let $\varepsilon_k \subseteqq E_{\aleph_0} \setminus 0$ be a subset, for which we suppose without loss of generality $\varepsilon_k = \{0, 1, \dots, k-1\}$.

Remark 3.3. The functions of the limit-logic have the property $f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$.

Theorem 3.2. *M is a maximal limit-logic.*

Proof. The limit-logic M being maximal follows immediately from the fact that M is increasing (according to Definition 2.7), a necessary and sufficient condition by Theorem 2.1 for the limit-logic M to be maximal.

4. The monotone function classes of the maximal limit-logic M

In this paragraph we examine the monotone classes of the maximal limit-logic M. We prove that the monotone classes of the M are similar to the monotone classes of P_k and their cardinality is continuum. Moreover, in the limit-logic M are continuum monotone pre-complete classes, while in the k-valued logic there is but a finite number of them.

Remark 4.1. Of course we shall speak about the elementwise restriction M_{ε_γ} of the function class to ε_γ to. This means no restriction as to the validity of the theorems. We define the set ε_γ, starting from a function $f(x_1, \dots, x_n) \in M$, as follows:

1. Let α denote the maximum in the range of

$$f(x_1, \dots, x_n).$$

2. Let β be the largest value, in some n-tuple, on which the function is not equal to 0. $(f(\dots, \beta, \dots) \neq 0)$.
3. Let γ max (α, β), $\varepsilon_\gamma = \{1, 2, \dots, \gamma\}$ $(\gamma \leqq k)$.

Definition 4.1. $\tilde{\alpha} \leqq \tilde{\beta}$, if $\alpha_i \leqq \beta_i$ where $\alpha_i, \beta_i \in E_k$ $(i=1, 2, \dots, n)$. We say that the function $f(x_1, \dots, x_n) \in P_k(P_{\aleph_0})$ is monotone with respect to the linear ordering if for an arbitrary pair $\tilde{\alpha}, \tilde{\beta} \in E_k$ $(E_{\aleph_0})$ $\tilde{\alpha} \leqq \tilde{\beta}$ implies $f(\tilde{\alpha}) \leqq f(\tilde{\beta})$.

Remark 4.2. Let $\prec$ denote the ordering $1 < 2 < 3 < \dots < n < n+1 < \dots$. Let be 0 a distinguished element in this ordering incomparable with any element of $E_{\aleph_0} \setminus 0$.

Denote by $\mathcal{M}^r_{\varepsilon_k}$ the set of the all monotone functions with respect to the order r over ε_k. Then let $\mathcal{M}^r = \bigcup_{k=2}^{\infty} \mathcal{M}^r_{\varepsilon_k}$.

We shall consider an arbitrary linear order and we shall prove, that the monotone functions with respect to this order are pre-complete, and different pre-complete classes belong to different linear orders.

Remark 4.3. Let $\mathcal{M}^r_{\varepsilon_k}$ be the set of all the functions $f(x_1, \dots, x_n)$ from the set $\mathcal{M}^r$ of monotone functions with respect to $\prec_r$ for which $f(x_1, \dots, e, \dots, x_n) = 0$ if $e \notin \varepsilon_k$ is true.

Theorem 4.1. *The function classes $\mathcal{M}^r_{\varepsilon_k}$ and $\mathcal{M}^r$ are closed.*

PROOF. Replace the variables by all the functions, which are monotone by $\prec_r$. Let

$$f_{\varepsilon_k}(x_1, x_2, \dots, x_n) = g_{\varepsilon_k}\big(g_{\varepsilon_k,1}(x_1, \dots, x_n), g_{\varepsilon_k,2}(x_1, \dots, x_n), \dots, g_{\varepsilon_k,m}(x_1, \dots, x_n)\big)$$

where the functions $g_{\varepsilon_k}, g_{\varepsilon_k,1}, \dots, g_{\varepsilon_k,m}$ are monotone according to $\prec_r$. We shall show that the function f is monotone according to $\prec_r$. Let us consider two sequences $\tilde{\alpha}$ and $\tilde{\beta}$, so that $\tilde{\alpha} \prec_r \tilde{\beta}$. It is clear that by the condition

$$g_{\varepsilon_k,i}(\tilde{\alpha}) \leqq_r g_{\varepsilon_k,i}(\tilde{\beta}), \quad (i = 1, 2, \dots, m)$$

holds, therefore the sequence

$$\{g_{\varepsilon_k,1}(\tilde{\alpha}),\ g_{\varepsilon_k,2}(\tilde{\alpha}), \dots, g_{\varepsilon_k,m}(\tilde{\alpha})\} \quad \text{and} \quad \{g_{\varepsilon_k,1}(\tilde{\beta}), g_{\varepsilon_k,2}(\tilde{\beta}), \dots, g_{\varepsilon_k,m}(\tilde{\beta})\}$$

are such that

$$\{g_{\varepsilon_k,1}(\tilde{\alpha}), g_{\varepsilon_k,2}(\tilde{\alpha}), \dots, g_{\varepsilon_k,m}(\tilde{\alpha})\} \leqq_r \{g_{\varepsilon_k,1}(\tilde{\beta}), g_{\varepsilon_k,2}(\tilde{\beta}), \dots, g_{\varepsilon_k,m}(\tilde{\beta})\}.$$

From here we have, because of the monotony of the function g_{ε_k}

$$f_{\varepsilon_k}(\tilde{\alpha}) = g_{\varepsilon_k}\big(g_{\varepsilon_k,1}(\tilde{\alpha}), \dots, g_{\varepsilon_k,m}(\tilde{\alpha})\big) \leqq_r g_{\varepsilon_k}\big(g_{\varepsilon_k,1}(\tilde{\beta}), \dots, g_{\varepsilon_k,m}(\tilde{\beta})\big) = f_\varepsilon(\tilde{\beta}).$$

So, this function class is invariant to superposition, therefore, it is a closed class. This completes the proof.

Denote by $C^j_{\varepsilon_k}(x)$ the next function:

$$\begin{array}{c|c|c|c} x & 0 & 1 \dots k & k+1 \dots \\ C^j_{\varepsilon_k} & 0 & j \dots j & 0 \;\; \dots \end{array}$$

Remark 4.4. If the order $\prec_r$ is the natural ordering $1<2<3<\dots$ we get the usual class $(\mathcal{M}^1)$ of monotone functions. The ideas of the proofs of the next few theorems and lemmas are due to Yablonski's paper [12].

Theorem 4.2. *The class of monotone functions is generated by the functions:* $\max_{\varepsilon_k}(x_1, x_2)$, $\min_{\varepsilon_k}(x_1, x_2)$, $c^j_{\varepsilon_k}(x)$ $(j=1, 2, \dots, k)$ *and the* $m^i_{\varepsilon_k}(x)$ $(i=2, \dots, k)$ *all of* $\mathcal{M}^1_{\varepsilon_k}$, *where*

$$m^i_{\varepsilon_k}(x) = \begin{cases} 1, & \textit{if} \quad 1 \leqq x < i \quad \textit{and} \quad x \in \varepsilon_k; \\ k, & \textit{if} \quad k \geqq x \geqq i \quad \textit{and} \quad x \in \varepsilon_k; \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Define the function $Z^{\tilde{\alpha},\beta}_{\varepsilon_k}(x_1, \dots, x_n)$ as follows:

$$Z^{\tilde{\alpha},\beta}_{\varepsilon_k}(x_1, \dots, x_n) = \begin{cases} \beta, & \text{if} \quad 1 \leqq x < i \quad \text{and} \quad x \in \varepsilon_k; \\ k, & \text{if} \quad k \geqq x \geqq i \quad \text{and} \quad x \in \varepsilon_k; \\ 0 & \text{otherwise}, \end{cases}$$

where $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \varepsilon_k$.

Obviously

$$Z^{\tilde{\alpha},\beta}_{\varepsilon_k}(x_1, \dots, x_n) = \min_{\varepsilon_k}\big(\beta, m^{\alpha_1}_{\varepsilon_k}(x_1), \dots, m^{\alpha_n}_{\varepsilon_k}(x_n)\big).$$

Let $Z_{\varepsilon_k}(x_1, \dots, x_n)$ be an arbitrary monotone function of $\mathcal{M}^1_{\varepsilon_k}$ then

$$Z_{\varepsilon_k}(x_1, \dots, x_n) = \max_{\tilde{\alpha}}{}_{\varepsilon_k} \{Z^{\tilde{\alpha}, f(\tilde{\alpha})}_{\varepsilon_k}(x_1, \dots, x_n)\}.$$

This completes the proof.

Lemma 4.1. *If the function* $Z(x_1, \dots, x_n) \in M_k$ *is not monotone, then with the help of the substitution of* C^j *we shall obtain the one-variable non:monote function.*

Proof. It will be sufficient to regard the elements from ε_k and so the proof is the same as in [12].

Definition 4.2. Let B be the sum of the sets $B_1, B_2, \dots, B_l$ so that: $B = B_1 + B_2 + \dots + B_l$, if

1. $B = B_1 \cup B_2 \cup \dots \cup B_l$.
2. B_i, B_j are pairwise disjoint, i.e. $B_i \cap B_j = \emptyset$, if $i \neq j$.

Lemma 4.1.1. *If the set is the direct product of two sets $\mathscr{C}$ and $\mathscr{D}$ which are non-empty and $\tilde{\gamma}^0 \in \mathscr{C}$, $\tilde{\delta}^0 \in \mathscr{D}$ then there are two elements $\tilde{\gamma} \in \mathscr{C}$ and $\tilde{\delta} \in \mathscr{D}$ such that they are next to each other by the ordering $\prec_r$. Moreover, if $\tilde{\gamma}^0 \prec_r \tilde{\delta}^0$, or, conversely, $\tilde{\gamma}^0 \succ_r \tilde{\delta}^0$, then $\tilde{\gamma}^0 \leqq_r \tilde{\gamma} \prec_r \tilde{\delta} \leqq_r \tilde{\delta}^0$ or $\tilde{\gamma}^0_r \geqq \tilde{\gamma}_r \succ \tilde{\delta}_r \geqq \tilde{\delta}^0$, respectively.*

PROOF. See [12].

Theorem 4.3. *The class of monotone functions $\mathscr{M}^1$ is pre-complete in the limit -logic M, where $\mathscr{M}^1$ is the class of monotone functions belonging to an arbitrary linear order.*

PROOF. We prove that $[\{f(x_1, \ldots, x_n)\} \cup \mathscr{M}^1] = M$, $f(x_1, \ldots, x_n) \overline{\in} \mathscr{M}^1$. Let $k \geqq \gamma$, where $\gamma = \max(\alpha, \beta)$; $\alpha = \max f(\tilde{x})$. Let β be the maximum of the values for which $f(\ldots, \beta, \ldots) \neq 0$. We show that the function $f_{\varepsilon_k}(x_1, \ldots, x_n)$ and the set $\mathscr{M}^1_{\varepsilon_k}$ generate the logic $[\{\mu_k(x_1, x_2)\}] = M_k$, which is isomorphic to the k-valued logic P_k.

Let $f_{\varepsilon_k}(x_1, \ldots, x_n) \overline{\in} \mathscr{M}^1_{\varepsilon_k}$. Then on account of Lemma 4.1 and with the help of the functions $C^j_{\varepsilon_k}$ we shall obtain the one-variable non-monotone function $g_{\varepsilon_k}(x)$. Suppose that over $x = t$

$$g_{\varepsilon_k}\{t) > g_{\varepsilon_k}(t+1) \quad (g(0) = 0)$$

$$h^1_{\varepsilon_k}(x) = \begin{cases} t, & \text{if} \quad x = 1; \\ t+1, & \text{if} \quad x \neq 1 \quad \text{and} \quad x \in \varepsilon_k; \\ 0 & \text{otherwise}; \end{cases}$$

$$h^2_{\varepsilon_k}(x) = \begin{cases} 1, & \text{if} \quad x \leqq g(t+1) \quad \text{and} \quad x \in \varepsilon_k; \\ k, & \text{if} \quad x > g(t+1) \quad \text{and} \quad x \in \varepsilon_k; \\ 0, & \text{if} \quad x \notin \varepsilon_k. \end{cases}$$

It is easy to see that $h^1_{\varepsilon_k}(x)$ and $h^2_{\varepsilon_k}(x)$ are monotone functions, that is, $h^1_{\varepsilon_k}(x)$, $h^2_{\varepsilon_k}(x) \in \mathscr{M}^1_{\varepsilon_k}$ and

$$h^2_{\varepsilon_k}\big(g_{\varepsilon_k}(h^1_{\varepsilon_k}(x))\big) = \begin{cases} k, & \text{if} \quad x = 1; \\ 1, & \text{if} \quad x \neq 1 \quad \text{and} \quad x \in \varepsilon_k; \\ 0 & \text{otherwise,} \end{cases}$$

where so $h^2_{\varepsilon_k}\big(g_{\varepsilon_k}(h^1_{\varepsilon_k}(x))\big) \equiv j^1_{\varepsilon_k}$ where

$$j^i_{\varepsilon_k}(x) = \begin{cases} k, & \text{if} \quad x = i \neq 0 \quad \text{and} \quad x \in \varepsilon_k; \\ 1, & \text{if} \quad x \neq i \qquad\quad \text{and} \quad x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the following functions:

$$\varphi^i_{\varepsilon_k}(x) = \begin{cases} 1, & \text{if} \quad x < i \quad \text{and} \quad x \in \varepsilon_k; \\ k, & \text{if} \quad x \geqq i \quad \text{and} \quad x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi^i_{\varepsilon_k}(x) = \begin{cases} 1, & \text{if} \quad x \leqq i \quad \text{and} \quad x \in \varepsilon_k; \\ 2, & \text{if} \quad x > i \quad \text{and} \quad x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\varphi^i_{\varepsilon_k}(x)$, $\psi^i_{\varepsilon_k}(x)\in\mathcal{M}^1_{\varepsilon_k}$. Thus we have the following functions:

$$C^j_{\varepsilon_k}(x),\ \max_{\varepsilon_k}(x_1, x_2),\ \min_{\varepsilon_k}(x_1, x_2) \quad\text{and}\quad j^i_{\varepsilon_k}(x) \quad (i = 1, 2, \ldots, k).$$

Lemma 4.3.1. *The set of functions* $C^j_{\varepsilon_k}(x)$, $\max_{\varepsilon_k}(x_1, x_2)$, $\min_{\varepsilon_k}(x_1, x_2)$ *is complete in* $M_k([\{\mu_k(x_1, x_2)\}]=M_k)$

$$C^j_{\varepsilon_k}(x) = \begin{cases} j & \textit{if} \quad x\in\varepsilon_k; \\ 0 & \textit{otherwise}. \end{cases}$$

$$j^i_{\varepsilon_k}(x) = \begin{cases} k, & \textit{if} \quad x = i \neq 0 \quad \textit{and} \quad x\in\varepsilon_k; \\ 1 & \textit{if} \quad x \neq i \quad\quad\ \textit{and} \quad x\in\varepsilon_k; \\ 0 & \textit{otherwise}. \end{cases}$$

PROOF. We shall obtain an arbitrary function from M_k from the given function system by induction on the number of the variables. Let $f(x_1, \ldots, x_n)\in M_k$ be an arbitrary function.

1. If f is 0-ary, then the statement is evident, since the initial system contains j $(j=1, \ldots, k)$.

2. Suppose we have constructed every n-variable function of the $\mathcal{M}^1_{\varepsilon_k}$ by superposition.

We show next that we can produce an arbitrary function of $n+1$ variables of $\mathcal{M}^1_{\varepsilon_k}$ and from the functions which were given in Lemma 4.3.1.

Therefore, we assume

$$\max_{\varepsilon_k}(y_1, y_2, \ldots, y_n) = \max_{\varepsilon_k}\{\max_{\varepsilon_k}[\ldots\max_{\varepsilon_k}(\max_{\varepsilon_k}(y_1, \ldots, y_n)]y_n)\}$$

(similarly for $\min_{\varepsilon_k}(y_1, y_2, \ldots, y_n)$).

Then

$$f_{\varepsilon_k}(x_1, \ldots, x_n, x_{n+1}) =$$

$$= \max_{\varepsilon_k}\{\min_{\varepsilon_k}[j^1_k(x_{n+1}), f(x_1, \ldots, x_n, 1)]\min_{\varepsilon_k}[j^2_{\varepsilon_k}(x_{n+1}), f(x_1, \ldots, x_n, 2)]\cdot$$

$$\cdot\min_{\varepsilon_k}[j^k_{\varepsilon_k}(x_{n+1}), f(x_1, \ldots, x_n, k)]\} \quad (\varepsilon_k = 1, 2, \ldots, k).$$

Hence we obtain the statement.

It follows from the foregoing that this system is complete. So the theorem is proved.

From the Theorem 4.3 follows

Theorem 4.4. *In the limit-logic M the monotone function class* $(\mathcal{M}^r)$ *belonging to any linear order is pre-complete.*

PROOF. If $\mathcal{M}^r_{\varepsilon_k}$ is the dual of the function class $\mathcal{M}^1_{\varepsilon_k}$ by the permutation

$$s_{\varepsilon_k}(x) = \begin{pmatrix} a_1, a_2, \ldots, a_k \\ 1,\ 2,\ \ldots, k \end{pmatrix},$$

then the results concerning the class $\mathcal{M}^1_{\varepsilon_k}$ can be extended (because of the duality principle) to the function class $\mathcal{M}^r_{\varepsilon_k}$.

Corollary 4.1. *Every function class* $\mathcal{M}^r_{\varepsilon_k}$ *is pre-complete in* M_k.

Theorem 4.5. *The cardinality of the pre-complete monotone classes is continuum in the limit-logic M.*

PROOF. Since in M there are countably many functions, so in M there cannot be more pre-complete classes than continuum. That is

a) the number of the different orders $\prec_r$ is continuum,

and

b) different $\mathcal{M}^r$-s correspond to different orders.

Corollary 4.2. *In the limit-logic M there are as many as continuum pre-complete classes.*

Summary. In this paper we prove, that the maximal limit-logic to be examined contains as many as continuum monotone pre-complete classes.

We obtained the following results:

1. In the maximal limit-logic M any monotone function class which belongs to a linear order r is pre-complete.

2. The number of pre-complete classes of monotone functions with respect to the linear ordering is continuum.

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