The generalized nilstufe of a group

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Abstract: The nilstufe of an abelian group G is the smallest positive integer n, if it exists, such that $R^{n+1}=0$ for every associative ring R with additive group G. A transfinite generalization of the nilstufe is given. Results concerning the nilstufe of G are carried over to the generalized nilstufe.

i. All groups considered here are assumed to be abelian, with addition the group operation, and all rings are assumed to be associative.

LEVITZKI [3] generalized the concept of nilpotence as follows: Let R be a ring. For every ordinal α define $R^{\alpha+1}=R^{\alpha}R$, for β a limit ordinal define $R^{\beta}=\bigcap_{\alpha<\beta} R^{\alpha}$. Call R a generalized nilpotent ring of degree α if $R^{\alpha}=0$ for some ordinal α, but $R^{\beta}\neq 0$ for all $\beta<\alpha$.

SZELE [4] defined the nilstufe of a group G to be the smallest positive integer n, if such exists, such that all rings with additive group G are nilpotent of degree less than or equal $n+1$. If no such positive integer exists, the nilstufe of G will be said to be ∞. In ii a generalization of Szele's nilstufe will be introduced which takes into consideration Levitzki's generalized nilpotence.

Notation:

1. G = a group.
2. R = a ring.
3. R^{+} = the additive group of R.
4. $\nu(G)$ = the nilstufe of G.
5. Z = the ring of integers.
6. Z_n = the ring of integers modulo a positive integer n.
7. Q = the field of rational numbers.
8. For $x\in G$, $T(x)$ = the type of x, [1, p. 109].
9. $r(G)$ = the rank of G, [1, p. 85].
10. Qe = a rank one torsion free group.
11. ω = the first infinite ordinal.
12. G_t = the torsion part of G.

Convention: $\alpha<\infty$ for every ordinal α.

ii. *Definition.* Let H be a subgroup of G. Define the generalized nilstufe of H in G, $n_G(H)$, to be the first ordinal, if such exists, such that $H^{n_G(H)}=0$ in every ring with additive group G. If no such ordinal exists, $n_G(H)=\infty$. The powers of H are to be understood in the same sense as the powers of R in the definition of generalized nilpotence. Call $n_G(G)$ the generalized nilstufe of G, and put $n_G(G)=n(G)$.

Example 1. Let $H \neq 0$ be a proper subgroup of Z^+. Then $n_Z(H) = \omega$.

Example 2. Let Qe_i be a rank one torsion free group with $T(e_i) = (i, i, \ldots, i, \ldots)$ for all $i \leqq \omega$. Put $G = \sum_{i<\infty} \oplus Qe_i$. Let R be an associative ring with $R^+ = G$. Every element in $(R^\omega)^+$ has type $(\infty, \infty, \ldots, \infty, \ldots)$ in G. However no non-zero element in G is of this type. Hence $R^\omega = 0$, and $n(G) \leqq \omega$. The products $e_i e_j = e_{i+j}$ for all $i, j < \omega$ induce a ring structure on G which is not nilpotent of finite degree. Therefore $n(G) = \omega$.

Observation. $n(G)$ is finite if and only if $\nu(G)$ is finite, in which case $n(G) = \nu(G) + 1$.

The following theorem generalizes [5, Theorem (a)], and will be proved similarly:

Theorem 2.1. *Let G be torsion free, and let H be a subgroup of G. Then either $n_G(H) = \infty$, or $n_G(H) \leqq r(G) + 1$. If moreover H is fully invariant in G then either $n_G(H) = \infty$, or $n_G(H) \leqq r(H) + 1$.*

Proof. Suppose that $n_G(H) < \infty$. Let $R^+ = G$, and suppose that $H^{r(G)+1} \neq 0$ in R. Put $A = Q \otimes_Z R$. Then A is a Q-algebra with $\dim_Q A = r(G)$. For every ordinal α, put B_α = the subalgebra of A generated by $Q \otimes_Z H^\alpha$ (the flatness of Q as a Z-module, allows for $(Q \otimes_Z H^\alpha)^+$ to be considered a subgroup of A^+). The sequence $B_1 \supset B_2 \supset \ldots \supset B_{r(G)+1} \neq 0$ is a properly descending chain of subalgebras of A. Hence $\dim_Q B_1 \geqq r(G) + 1 > \dim_Q A$, a contradiction.

If H is fully invariant in G, then H is an ideal in R, and $\dim_Q B_1 = r(H)$. However, the proper descent of the chain $B_1 \supset B_2 \supset \ldots \supset B_{r(H)+1} \neq 0$ implies that $\dim_Q B_1 \geqq r(H) + 1$, a contradiction.

Gardner [2, p. 5] introduced the following ascending chain of fully invariant subgroups of G:

$G(1)$ = the absolute annihilator of G, [1, Problem 94].

$G(\alpha+1)/G(\alpha)$ = the absolute annihilator of $G/G(\alpha)$ for every ordinal α.

For β a limit ordinal, $G(\beta) = \bigcup_{\alpha<\beta} G(\alpha)$.

As was shown in case the above chain is finite [2, Corollary 2.4], we have:

Theorem 2.2. *If $G = G(\alpha)$ for some ordinal α, then $n(G) \leqq \alpha + 1$.*

Proof. It suffices to show that $G(\beta)$ annihilates R^β for every ring R with $R^+ = G$. This is true for $\beta = 1$ by the definition of $G(1)$. Let $\beta > 1$, and suppose that $G(\alpha)$ annihilates R^α for all $\alpha < \beta$ and all rings R with $R^+ = G$.

1) Let $\beta = \alpha + 1$. Then $R^\beta \cdot G(\beta) = R^\alpha \cdot R \cdot G(\alpha+1) \subseteqq R^\alpha \cdot G(\alpha) = 0$.

2) Let β be a limit ordinal, and let $x \in G(\beta)$. Then $x \in G(\alpha)$ for some $\alpha < \beta$. Hence $R^\beta \cdot x \subseteqq R^\alpha \cdot x = 0$.

Szele [1, Theorem 120.3] has shown that for G a torsion group, either $\nu(G) = 1$, or $\nu(G) = \infty$. The following is a generalization of this result:

Theorem 2.3. *Let G be a torsion group. Then either $n(G) = 2$, or $n(G) = \infty$.*

PROOF. If G is divisible, then $n(G)=2$, [1, Theorem 120.3]. Otherwise $G \cong Z_n^+ \oplus K$, $n>1$, [1, Corollary 27.3]. Let S be the zeroring on K. The ring direct sum $R=Z_n \oplus S$ is clearly not generalized nilpotent. Hence $n(G)=\infty$.

Theorem 2.4. $n(G)<\infty$ *if and only if* G_t *is divisible, and* $n(G/G_t)<\infty$, *in which case* $n(G) \leqq 2n(G/G_t)$.

PROOF. 1) Suppose that $n(G)<\infty$. If G_t is not divisible, then the argument used in the proof of Theorem 2.3 shows that $n(G)=\infty$, a contradiction. Hence G_t is divisible, and so $G \cong G_t \oplus (G/G_t)$. This clearly implies that $n(G/G_t)<\infty$.

2) Suppose that G_t is divisible, and that $n(G/G_t)=\alpha<\infty$. Let R be a ring with $R^+=G$. Since G_t is an ideal in R, we have that $(R/G_t)^\alpha=0$, or $R^\alpha \subseteqq G_t$. By [1, Theorem 120.3] $G_t^2=0$, so that $R^{2\alpha}=0$.

Problem: SZELE [4] has shown that for every positive integer n, there exists a group G with $\nu(G)=n$. Is it true that for every ordinal $\alpha>1$, there exists a group G with $n(G)=\alpha$?

References

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