On homotopy classes of mappings into flat Riemannian manifolds

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Introduction

To every C^2-mapping of a Riemannian manifold into another it can be assigned a vector valued 1-form with values in a vector bundle obtained by pulling back the tangent bundle of the target manifold along this mapping. Vector bundles obtained in this way from homotopic mappings can be canonically identified by means of parallel displacement. So, the theory of such vector valued forms gives a useful tool for the study of homotopy classes of mappings. Therefore the first paragraph of this note is devoted to the elementary theory of vector valued forms. Although the de Rham decomposition theorem is valid for vector valued forms, the spaces of exact and coexact forms are not necessarily orthogonal to each other. Avoiding this difficulty, in the second paragraph of this note we deal with the case when the target manifold is flat. Then the usual identity $d^2=0$ is valid for the operator d of exterior differentiation. We give a simple proof for the Eells—Sampson's homotopy theorem of [1] in the special case when the target manifold is flat, by means of reasonings essentially different from the original. In the last part of this paragraph we examine properties of homotopies between harmonic mappings. In the third paragraph we define the family of homotopy classes $[M, M']$ of mappings of a Riemannian manifold M into another M'. This family depends only on the homotopy types of the manifolds M and M'. If $M'=G$ is a Lie group then $[M, G]$ can be endowed with a group structure. By means of a factorization theorem of Lichnerowicz in [5] we obtain that $[M, G]\cong \mathbf{Z}^{b_1(M)\cdot b_1(G)}$ holds for any commutative group G where b_1 denotes the first Betti number.

I. Differential operators on a Riemannian-connected bundle

Let $W\to M$ be a vector bundle over a Riemannian manifold M and denote $\mathbf{T}^{[p]}(M)\to M, p\in\mathbf{N}$, the bundle of p-covectors of M. Further put

$$\Lambda^0(M,W)=\operatorname{Sec} W$$

and, for each $p\in\mathbf{N}$, put

$$\Lambda^p(M,W)=\operatorname{Sec}(W\otimes \mathbf{T}^{[p]}(M)).$$

The elements of $\Lambda^p(M, W)$ are called p-forms on M with values in W.

A covariant differentiation on the vector bundle $W \to M$ is a linear mapping

$$\nabla\colon \Lambda^0(M, W) \to \Lambda^1(M, W)$$

which satisfies the derivation rule

$$\nabla(\mu w) = w \otimes d\mu + \mu \nabla w$$

for $w \in \Lambda^0(M, W)$ and for every scalar μ on M.

The operator ∇ defines a covariant differentiation

$$\iota_X \circ \nabla = \nabla_X\colon \Lambda^0(M, W) \to \Lambda^0(M, W)$$

for every vector field $X \in \mathfrak{X}(M)$ on M. It can be canonically extended to a covariant differentiation

$$\nabla_X\colon \Lambda^p(M, W) \to \Lambda^p(M, W)$$

of p-forms, $p \in \mathbf{N}$, by the agreement

$$\nabla_X(w \otimes \lambda) = (\nabla_X w) \otimes \lambda + w \otimes (\nabla_X \lambda).$$

A covariant differentiation ∇ defines a mapping

$$\Pi\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \operatorname{Sec} W \to \operatorname{Sec} W$$

by

$$\Pi(X, Y) w = \nabla_X \nabla_Y w - \nabla_Y \nabla_X w - \nabla_{[X, Y]} w,$$

where $X, Y \in \mathfrak{X}(M)$ and $w \in \operatorname{Sec} W$. This mapping Π is the curvature of the vector bundle $W \to M$.

From now on we consider a fixed covariant differentiation ∇ on the vector bundle $W \to M$.

The exterior differentiation of the vector bundle $W \to M$ is a linear mapping

$$d\colon \Lambda^p(M, W) \to \Lambda^{p+1}(M, W) \quad p \in \mathbf{N} \quad \text{or} \quad p = 0$$

for which

$$d(w \otimes \lambda) = (\nabla w) \wedge \lambda + w \otimes (d\lambda)$$

holds for every $w \in \operatorname{Sec} W$ and for every p-form $\lambda \in \Lambda^p(M)$, where $d\lambda$ is the usual exterior differentiation of the p-form λ. Obviously $d = \nabla$ if $p = 0$.

Now let us suppose that the bundle $W \to M$ is Riemannian-connected, i.e. each of the fibres has a positive-definite inner product, which we shall denote by $(\, , \,)$, and the covariant differentiation ∇ preserves the metric on the fibres of W, i.e. for each $w, w' \in \operatorname{Sec} W$ and $X \in \mathfrak{X}(M)$

$$\nabla_X(w, w') = (\nabla_X w, w') + (w, \nabla_X w')$$

holds. This inner product can be extended to an inner product of the bundle $\Lambda^p(M, W) \to M$ by the agreement

$$(w \otimes \lambda, w' \otimes \lambda') = (w, w')(\lambda, \lambda')$$

for each $w, w' \in \operatorname{Sec} W$ and $\lambda, \lambda' \in \Lambda^p(M)$, where (λ, λ') is the usual inner product of the forms λ and λ'.

From now on, throughout this note, we shall always assume M to be compact and oriented and we denote its volume element by $v \in \Lambda^n(M)$, $n = \dim M$.

The global scalar product of the p-forms Φ, $\Psi \in \Lambda^p(M, W)$ as usual is defined by

$$\langle \Phi, \Psi \rangle = \int_M (\Phi, \Psi) v.$$

Let

$$\partial\colon \Lambda^p(M, W) \to \Lambda^{p-1}(M, W) \quad p \in \mathbf{N}$$

be the adjoint operator of d with respect to the global scalar product and put $\partial = 0$ if $p = 0$. Then for each $\Phi \in \Lambda^p(M, W)$ and $\Psi \in \Lambda^{p+1}(M, W)$

$$\langle d\Phi, \Psi \rangle = \langle \Phi, \partial\Psi \rangle$$

holds. Finally let

$$\Delta = d \circ \partial + \partial \circ d\colon \Lambda(M, W) \to \Lambda(M, W) \quad p \in \mathbf{N} \quad \text{or} \quad p = 0$$

be the Laplace operator of the Riemannian vector bundle $W \to M$. A p-form Ω on M with values in W is said to be harmonic if $\Delta\Omega = 0$. It is known that there is a decomposition

$$\Lambda^p(M, W) = H^p(M, W) \oplus \big(d\Lambda^{p-1}(M, W) + \partial\Lambda^{p+1}(M, W)\big), \quad p \in \mathbf{N}$$

where $H^p(M, W)$, the space of harmonic p-forms with values in W, is orthogonal to the other two summands [1].

Our present aim in this paragraph is to prove the following explicite formula:

$$\partial\Phi = -\operatorname{trace}\{(X, Y) \to \nabla_X \circ \iota_Y \Phi\} = -\operatorname{trace}\{(X, Y) \to \iota_X \circ \nabla_Y \Phi\}$$

for every $\Phi \in \Lambda^p(M, W)$.

We need the following three lemmas:

Lemma (1.1). *Let X and Y be vector fields on M such that $\nabla_Y X = 0$ holds. Then ι_X and ∇_Y commute on $\Lambda^p(M, W)$ for each $p \in \mathbf{N}$ or $p = 0$.*

Proof. At first we prove that ι_X and ∇_Y commute on $\Lambda^p(M)$. Indeed, if $\lambda \in \Lambda^p(M)$ and $X_2, \dots, X_p$ are vector fields on M then we have

$$\big((\iota_X \circ \nabla_Y)\lambda\big)(X_2, \dots, X_p) = p(\nabla_Y \lambda)(X, X_2, \dots, X_p) = pY\big(\lambda(X, X_2, \dots, X_p)\big) -$$

$$-p\lambda(\nabla_Y X, X_2, \dots, X_p) - \sum_{i=2}^{p} p\lambda(X, X_2, \dots, X_{i-1}, \nabla_Y X_i, X_{i+1}, \dots, X_p) =$$

$$= \big((\nabla_Y \circ \iota_X)\lambda\big)(X_2, \dots, X_p).$$

Turning to the general case, we may restrict ourselves to decomposable p-forms on M with values in W, $\Phi = w \otimes \lambda$, where $w \in \operatorname{Sec} W$ and $\lambda \in \Lambda^p(M)$. Then

$$(\nabla_Y \circ \iota_X)(w \otimes \lambda) = \nabla_Y(w \otimes \iota_X \lambda) = (\nabla_Y w) \otimes (\iota_X \lambda) + w \otimes (\nabla_Y \circ \iota_X \lambda) =$$

$$= (\nabla_Y w) \otimes (\iota_X \lambda) + w \otimes (\iota_X \circ \nabla_Y \lambda) = \iota_X\big((\nabla_Y w) \otimes \lambda + w \otimes (\nabla_Y \lambda)\big) = (\iota_X \circ \nabla_Y)(w \otimes \lambda),$$

which accomplishes the proof.

Corollary (1.1). $\operatorname{trace}\{(X, Y) \to \nabla_X \circ \iota_Y \Phi\} = \operatorname{trace}\{(X, Y) \to \iota_X \circ \nabla_Y \Phi\}$.

PROOF. Let $m\in M$ be an arbitrary point and let $e_1, \dots, e_n$ be a frame in $T_m(M)$. Extend them to vector fields $E_1, \dots, E_n$ such that $g(E_i, E_j)(m)=\delta_{ij}$ and $\nabla_{E_i}E_j(m)=0$ for each $i, j=1, \dots, n$. Then we have

$$\operatorname{trace}\{(X, Y)\to\nabla_X\circ\iota_Y\Phi\}_m = \sum_{i=1}^n \nabla_{E_i}\circ\iota_{E_i}\Phi(m) = \sum_{i=1}^n \iota_{E_i}\circ\nabla_{E_i}\Phi(m) =$$

$$= \operatorname{trace}\{(X, Y)\to\iota_X\circ\nabla_Y\Phi\}_m.$$

Lemma (1.2). *Let $\varrho\in\Lambda^1(M)$ be an arbitrary 1-form on M and let us consider the $(n-1)$-form*

$$\alpha = \operatorname{trace}\{(X, Y)\to(\iota_X\varrho)\iota_Y v\}\in\Lambda^{n-1}(M).$$

Then

$$d\alpha = \operatorname{trace}\{(X, Y)\to\nabla_Y((\iota_X\varrho)v)\}.$$

PROOF. We have

$$d\alpha = d\operatorname{trace}\{(X, Y)\to(\iota_X\varrho)\iota_Y v\} = \operatorname{trace}\{(X, Y)\to d((\iota_X\varrho)\iota_Y v)\} =$$

$$= \operatorname{trace}\{(X, Y)\to(d\circ\iota_Y)(\iota_X\varrho)v\} = \operatorname{trace}\{(X, Y)\to(\iota_Y\circ d+d\circ\iota_Y)((\iota_X\varrho)v)\} =$$

$$= \operatorname{trace}\{(X, Y)\to L_Y((\iota_X\varrho)v)\} = \operatorname{trace}\{(X, Y)\to(L_Y(\iota_X\varrho))v\} =$$

$$= \operatorname{trace}\{(X, Y)\to(\nabla_Y(\iota_X\varrho))v\} = \operatorname{trace}\{(X, Y)\to\nabla_Y((\iota_X\varrho)v)\}.$$

The metric tensor g of the Riemannian manifold M yields a canonical isomorphism

$$\gamma\colon \mathfrak{X}(M)\to\Lambda^1(M)$$

by $\gamma(X)Y=g(X, Y)$ for every $X, Y\in\mathfrak{X}(M)$.

Lemma (1.3). $d\Phi=(-1)^p \operatorname{trace}\{(X, Y)\to(\nabla_X\Phi)\wedge\gamma(Y)\}$ *holds for every* $\Phi\in\Lambda^p(M, W)$.

PROOF. As in the proof of Lemma (1.1), we restrict ourselves to decomposable p-forms on M with values in W, $\Phi=w\otimes\lambda$, where $w\in\operatorname{Sec} W$ and $\lambda\in\Lambda^p(M)$. Then

$$(\nabla_X(w\otimes\lambda))\wedge\gamma(Y) = (\nabla_X w)\otimes(\lambda\wedge\gamma(Y))+w\otimes((\nabla_X\lambda)\wedge\gamma(Y))$$

and

$$d(w\otimes\lambda) = (\nabla w)\wedge\lambda+w\otimes d\lambda$$

are valid and therefore it is sufficient to prove the formulas:

(a) $$(\nabla w)\wedge\lambda = (-1)^p\operatorname{trace}\{(X, Y)\to(\iota_X\circ\nabla w)\otimes(\lambda\wedge\gamma(Y))\},$$

(b) $$d\lambda = \operatorname{trace}\{(X, Y)\to\gamma(Y)\wedge(\nabla_X\lambda)\}.$$

We verify formula (a) in case when ∇w is decomposable, i.e. $\nabla w=v\otimes\omega$ holds for some $v\in\operatorname{Sec} W$ and $\omega\in\Lambda^1(M)$. Let $m\in M$ be an arbitrary point and choose vector fields $E_1, \dots, E_n$ as in the proof of Lemma (1.1). Then

$$(\nabla w)\wedge\lambda(m) = v\otimes(\omega\wedge\lambda)(m) = v\otimes\left(\sum_{i=1}^{n}\omega(E_i)\gamma(E_i)\right)\wedge\lambda(m) =$$

$$= \sum_{i=1}^{n}\omega(E_i)v\otimes(\gamma(E_i)\wedge\lambda)(m) = (-1)^p\sum_{i=1}^{n}\omega(E_i)v\otimes(\lambda\wedge\gamma(E_i))(m) =$$

$$= (-1)^p\,\text{trace}\,\{(X, Y)\to(\iota_X\circ\nabla w)\otimes(\lambda\wedge\gamma(Y)).$$

Turning to the proof of formula (b), let $m\in M$ and let $E_1, \dots, E_n$ be as above. Since $[E_i, E_j](m)=0$ for every $i, j=1, \dots, n$ we have

$$(d\lambda)(E_{i_0}, \dots, E_{i_p})(m) = \frac{1}{(p+1)}\sum_{j=0}^{p}(-1)^j E_{i_j}(\lambda(E_{i_0}, \dots, \hat{E}_{i_j}, \dots, E_{i_p}))(m) =$$

$$= \frac{1}{(p+1)}\sum_{j=0}^{p}(-1)^j(\nabla_{E_{i_j}}\lambda)(E_{i_0}, \dots, \hat{E}_{i_j}, \dots, E_{i_p})(m) =$$

$$= \frac{1}{(p+1)}\sum_{j=0}^{p}(-1)^j\gamma(E_{i_j})(E_{i_j})(m)(\nabla_{E_{i_j}}\lambda)(E_{i_0}, \dots, \hat{E}_{i_j}, \dots, E_{i_p})(m) =$$

$$= \sum_{i=1}^{n}(\gamma(E_i)\wedge(\nabla_{E_i}\lambda))(E_{i_0}, \dots, E_{i_p})(m) = \text{trace}\,\{(X, Y)\to\gamma(Y)\wedge(\nabla_X\lambda)\}(E_{i_0}, \dots, E_{i_p})(m)$$

which accomplishes the proof.

Now we are able to verify the previously mentioned formula of the adjoint operator ∂ as follows:

Proposition (1.1). $\partial\Phi = -\text{trace}\,\{(X, Y)\to\iota_X\circ\nabla_Y\Phi\} = -\text{trace}\,\{(X, Y)\to\nabla_X\circ\iota_Y\Phi\}$ *is valid for every* $\Phi\in\Lambda^p(M, W)$.

Proof. By virtue of Corollary (1.1) it is enough to prove the first equality. The adjoint operator of d is unique and therefore we have to check the validity of the characteristic formula of the adjoint operator ∂, i.e. restricting ourselves to decomposable forms, we have to show that

$$\langle v\otimes\omega, d(w\otimes\lambda)\rangle = \langle -\text{trace}\,\{(X, Y)\to(\iota_Y\circ\nabla_X)(v\otimes\omega)\}, w\otimes\lambda\rangle$$

is valid for every $v, w\in\text{Sec}\,W$ and $\lambda\in\Lambda^p(M)$, $\omega\in\Lambda^{p+1}(M)$. We have

$$((\iota_Y\circ\nabla_X)(v\otimes\omega), w\otimes\lambda) = ((\nabla_X v)\otimes(\iota_Y\omega), w\otimes\lambda) + (v\otimes(\iota_Y\circ\nabla_X)\omega, w\otimes\lambda) =$$

$$= (\nabla_X v, w)(\iota_Y\omega, \lambda) + (v, w)(\iota_Y(\nabla_X\omega), \lambda) = (\nabla_X v, w)(\omega, \gamma(Y)\wedge\lambda) +$$

$$+ (v, w)(\nabla_X\omega, \gamma(Y)\wedge\lambda) = (\omega, \gamma(Y)\wedge\lambda)X(v, w) - (v, \nabla_X w)(\omega, \gamma(Y)\wedge\lambda) +$$

$$+ (v, w)X(\omega, \gamma(Y)\wedge\lambda) - (v, w)(\omega, \nabla_X(\gamma(Y)\wedge\lambda)) = X((v, w)(\omega, \gamma(Y)\wedge\lambda)) -$$

$$-(v\otimes\omega, \nabla_X(w\otimes(\gamma(Y)\wedge\lambda))) = X(v\otimes\omega, w\otimes(\gamma(Y)\wedge\lambda)) -$$

$$-(v\otimes\omega, \nabla_X(w\otimes(\gamma(Y)\wedge\lambda))).$$

Now we state that

$$\operatorname{trace}\left\{(X, Y) \to \int_M X(v\otimes\omega, \gamma(Y)\wedge\lambda)v\right\} = 0.$$

Using the definition of the inner product on $\Lambda^{p+1}(M, W)\to M$ we obtain

$$(v\otimes\omega, w\otimes(\gamma(Y)\wedge\lambda)) = (v, w)(\omega, \gamma(Y)\wedge\lambda) = (v, w)(\imath_Y\omega, \lambda) = (\imath_Y\omega, (v, w)\lambda)$$

and therefore it is enough to prove that

$$\operatorname{trace}\left\{(X, Y) \to \int_M X(\imath_Y\mu, v)v\right\} = \operatorname{trace}\left\{(X, Y) \to \int_M \nabla_X((\imath_Y\mu, v)v)\right\} = 0$$

is valid for every $\mu\in\Lambda^{p+1}(M)$ and $v\in\Lambda^p(M)$.

Let $\varrho\in\Lambda^1(M)$ be the 1-form on M defined by

$$\varrho(X) = (\imath_X\mu, v), \quad X\in\mathfrak{X}(M).$$

Defining the $(n-1)$-form α according to Lemma (1.2) we have, by virtue of Stokes theorem, that

$$\operatorname{trace}\left\{(X, Y) \to \int_M \nabla_X((\imath_Y\mu, v)v)\right\} = \int_M d\alpha = 0.$$

So, using Lemma (1.3) we obtain

$$\begin{aligned}\langle v\otimes\omega, d(w\otimes\lambda)\rangle &= \langle v\otimes\omega,\ \operatorname{trace}\{(X, Y) \to \nabla_X(w\otimes(\gamma(Y)\wedge\lambda))\}\rangle = \\ &= \operatorname{trace}\{(X, Y) \to \langle v\otimes\omega, \nabla_X(w\otimes(\gamma(Y)\wedge\lambda))\rangle\} = \\ &= -\operatorname{trace}\{(X, Y) \to \langle(\imath_Y\circ\nabla_X)(v\otimes\omega), w\otimes\lambda\rangle\} = \\ &= \langle -\operatorname{trace}\{(X, Y) \to (\imath_Y\circ\nabla_X)(v\otimes\omega)\}, w\otimes\lambda\rangle\end{aligned}$$

which accomplishes the proof.

II. Homotopy classes and harmonic mappings

Let M and M' denote complete Riemannian manifolds of dimension n and k respectively, and suppose further that M is compact and oriented. If $f\colon M\to M'$ is a mapping of class C^2 then let $F\to M$ be the vector bundle obtained by pulling back the tangent bundle $TM'\to M'$ along f. Then the elements of $\Lambda^0(M, F)$ are canonically identified with the vector fields along f and the tangent map f_* can be considered as a specific 1-form on M with values in F. The covariant differentiation ∇' of M' canonically determines a covariant differentiation

$$\nabla^F\colon \Lambda^0(M, F) \to \Lambda^1(M, F)$$

by

$$\imath_X(\nabla^F u) = \nabla^F_X u = \nabla'_{f_*X} u\in\Lambda^0(M, F)$$

for each $X\in\mathfrak{X}(M)$ and $u\in\Lambda^0(M, F)$. The metric tensor g' of M' canonically induces a positive-definite inner product on the fibres of $F\to M$ and so the bundle $F\to M$ becomes a Riemannian-connected bundle. From now on, if there is no danger of confusion, we denote the differential operators on $F\to M$ simply by ∇, d, ∂ and $\Delta=d\circ\partial+\partial\circ d$ omitting the upper index F.

The mapping $f\colon M\to M'$ of class C^2 is said to be harmonic if $\partial f_*=0$.

Lemma (2.1). *f_* is closed, i.e. $df_*=0$. The mapping $f\colon M\to M'$ is harmonic if and only if $\Delta f_*=0$.*

PROOF. It is enough to restrict ourselves to the case when $f_*=w\otimes\omega$, where $w\in\operatorname{Sec} F$ and $\omega\in\Lambda^1(M)$. From the definition of the tangent map f_* it follows that ω is exact, i.e. $\omega=d\mu$ is valid for some scalar μ on M. Then we have

$$df_* = d(w\otimes d\mu) = (\nabla w)\wedge(d\mu)+w\otimes(d^2\mu) = (\nabla w)\wedge(d\mu).$$

So, it is sufficient to prove that the form $(\nabla w)\wedge(d\mu)$ is symmetric. Indeed, if X and Y are vector fields on M then

$$\iota_Y\circ\iota_X\big((\nabla w)\wedge(d\mu)\big) = \frac{1}{2}\,\iota_Y\circ\iota_X\big((\nabla w)\otimes(d\mu)-(d\mu)\otimes(\nabla w)\big) =$$

$$= \frac{1}{2}\big(d\mu(Y)\nabla_X w-d\mu(X)\nabla_Y w\big) = \frac{1}{2}\big(d\mu(Y)\nabla'_{f_*(X)}w-d\mu(X)\nabla'_{f_*(X)}w\big) =$$

$$= \frac{1}{2}\big(d\mu(Y)\,\nabla'_{d\mu(X)w}w-d\mu(X)\,\nabla'_{d\mu(Y)w}w\big)=\iota_X\circ\iota_Y\big((\nabla w)\wedge(d\mu)\big),$$

which accomplishes the proof of the first part of the lemma.

Now, suppose that $\Delta f_*=(d\circ\partial)f_*=0$. Then

$$0 = \langle\Delta f_*, f_*\rangle = \langle(d\circ\partial)f_*, f_*\rangle = \langle\partial f_*, \partial f_*\rangle$$

and therefore f is harmonic. The converse is trivial and thus our lemma is proved.

Now let us consider the de Rham decomposition

$$f_* = du+\partial\lambda+\Omega,$$

where $u\in\Lambda^0(M, F)$ is a vector field along f, $\lambda\in\Lambda^2(M, F)$ and Ω is a harmonic 1-form on M with values in F. Our present aim is to eliminate the second term of the right hand side. For this reason we calculate the operator d^2 explicitly as follows:

Lemma (2.2). *Let $W\to M$ be a Riemannian-connected bundle on M and $m\in M$. Let $e_1, \dots, e_n$ be a frame in $T_m(M)$ with vector field extensions $E_1, \dots, E_n$ such that $g(E_i, E_j)(m)=\delta_{ij}$ and $\nabla_{E_i}E_j(m)=0$ for each $i, j=1, \dots, n$. Then*

$$d^2w(m) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Pi(E_i, E_j)w\otimes\gamma(E_i)\wedge\gamma(E_j)(m).$$

PROOF. Using Lemma (1.3) we have

$$d^2w(m) = \sum_{j=1}^{n} \nabla_{E_j}\left\{\sum_{i=1}^{n} (\nabla_{E_i} w)\otimes\gamma(E_i)\right\}\wedge\gamma(E_j)(m) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} (\nabla_{E_j}\nabla_{E_i} w)\otimes\gamma(E_i)\wedge\gamma(E_j)(m) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \{(\nabla_{E_j}\nabla_{E_i} - \nabla_{E_i}\nabla_{E_j} - \nabla_{[E_i, E_j]})w\}\otimes\gamma(E_i)\wedge\gamma(E_j)(m) +$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} (\nabla_{E_i}\nabla_{E_j} w)\otimes\gamma(E_i)\wedge\gamma(E_j)(m) =$$

$$= - \sum_{i=1}^{n}\sum_{j=1}^{n} \Pi(E_i, E_j)w\otimes\gamma(E_i)\wedge\gamma(E_j)(m) - d^2w(m),$$

and hence our formula for d^2w is obtained.

From Lemma (2.2), it follows easily that $d^2=0$ if the target manifold M' is flat Riemannian manifold.

Corollary (2.1). *Let $f\colon M\to M'$ be a mapping of class C^2 where M' is a flat Riemannian manifold. Then the decomposition*

$$f_* = du + \Omega$$

holds, where $u\in \text{Sec}\, F$ is a vector field along f and Ω is a harmonic 1-form on M with values in F.

PROOF. For the 1-form $f_*\in A^1(M, F)$ the de Rham decomposition

$$f_* = du + \partial\lambda + \Omega$$

is valid, where $u\in \text{Sec}\, F$, $\lambda\in A^2(M, F)$ and Ω is a harmonic 1-form on M with values in F. Applying the operator d to each side of the above equation and using Lemma (2.1) we obtain

$$0 = df_* = d^2u + d\partial\lambda + d\Omega = d\partial\lambda.$$

Hence $0=\langle(d\circ\partial)\lambda, \lambda\rangle=\langle\partial\lambda, \partial\lambda\rangle$, i.e. $\lambda=0$, which accomplishes the proof.

Now let $f\colon M\to M'$ be a mapping of class C^2 and let w be an arbitrary vector field along f. Since M' is complete we can define a mapping

$$h\colon M \to M'$$

by

$$h(m) = \exp'_{f(m)} w_m, \quad m\in M.$$

Let $H\to M$ be the vector bundle over M obtained by pulling back the tangent bundle $TM'\to M'$ along h. So $F\to M$ and $H\to M$ are vector bundles over the common Riemannian manifold M.

Let

$$\tau_w\colon \text{Sec}\, F \to \text{Sec}\, H$$

be the canonical isomorphism defined by means of parallel displacement, i.e. if

$u \in \mathrm{Sec}\, F$ is a vector field along f and if $m \in M$ then let $\tau_w(u)_m \in T_{h(m)}(M')$ obtained by the parallel displacement of u along the geodesic from $f(m)$ to $h(m)$

$$s \mapsto \exp_{f(m)}(s \cdot w_m), \quad 0 \leqq s \leqq 1.$$

It is clear that τ_w is an isomorphism between the vector spaces Sec F and Sec H. The mapping

$$\tau_w\colon \Lambda^0(M, F) \to \Lambda^0(M, H)$$

can be extended to an isomorphism

$$\tau_w\colon \Lambda^p(M, F) \to \Lambda^p(M, H)$$

for every $p \in \mathbf{N}$ by the agreement

$$\tau_w(u \otimes \lambda) = \big(\tau_w(u)\big) \otimes \lambda$$

for each $u \in \mathrm{Sec}\, F$ and $\lambda \in \Lambda^p(M)$. Furthermore τ_w commutes with ι_X.

Lemma (2.3). *The mappings f and h are homotopic.*

PROOF. Let

$$\Phi\colon [0, 1] \times M \to M'$$

be defined by

$$\Phi(s, m) = \exp_{f(m)}(sw_m), \quad 0 \leqq s \leqq 1 \quad \text{and} \quad m \in M.$$

Obviously Φ is a homotopy between the mappings f and h.

The isomorphism τ_w ensures a comparison between the vector bundles $F \to M$ and $H \to M$. Our present aim is to give explicite connections between the covariant differentiations ∇^F and ∇^H of the bundles $F \to M$ and $H \to M$, respectively.

In the proof of lemma below we use the following notation: If $\alpha\colon (a, b) \to M'$ is a curve of class C^2 and $a \leqq x \leqq y \leqq b$ then

$$\tau_\alpha(x, y)\colon T_{\alpha(x)}(M') \to T_{\alpha(y)}(M')$$

denotes the parallel displacement along α from $\alpha(x)$ to $\alpha(y)$. If $x=0$ and $y=1$ then we use τ_α instead of $\tau_\alpha(0, 1)$.

Lemma (2.4). *Let M' be flat Riemannian manifold. Then the diagram*

$$\begin{array}{ccc} \mathrm{Sec}\, F & \xrightarrow{\tau_w} & \mathrm{Sec}\, H \\ \downarrow{\scriptstyle \nabla_X^F} & & \downarrow{\scriptstyle \nabla_X^H} \\ \mathrm{Sec}\, F & \xrightarrow{\tau_w} & \mathrm{Sec}\, H \end{array}$$

commutes for every $X \in \mathfrak{X}(M)$.

PROOF. Let $m \in M$ and $\gamma\colon (-\varepsilon, \varepsilon) \to M$ a curve of class C^∞ with $\gamma(0)=m$ and $\dot{\gamma}(0)=X_m$. Put $\varphi=f\circ\gamma$ and $\psi=h\circ\gamma$. Then $w\circ\gamma$ is a vector field along φ. If $u \in \mathrm{Sec}\, F$ is an arbitrary vector field along f then

$$(\tau_w \circ \nabla_X^F u)_m = \tau_w \circ \nabla_{\dot{\varphi}(0)} u = \tau_w \lim_{h \to 0} \frac{1}{h} \{\tau_\varphi\{0, h)^{-1} u_{\varphi(h)} - u_{\varphi(0)}\} =$$

$$= \lim_{h \to 0} \frac{1}{h} \{\tau_w \circ \tau_\varphi(0, h)^{-1} u_{\varphi(h)} - \tau_w u_{\varphi(0)}\}.$$

On the other hand

$$(\nabla_X^H \circ \tau_w u)_m = \nabla_{\dot{\psi}(0)} \tau_w(u) = \lim_{h\to 0} \frac{1}{h} \{\tau_\psi(0, h)^{-1} \circ \tau_w \circ u_{\varphi(h)} - \tau_w u_{\varphi(0)}\}.$$

Since M' is flat the parallel displacement depends only upon the homotopy classes of curves and therefore

$$\tau_\psi(0, h)^{-1} \circ \tau_w = \tau_w \circ \tau_\varphi(0, h)^{-1}, \quad -\varepsilon < h < \varepsilon$$

which accomplishes the proof.

Lemma (2.5). *If M' is flat then $h_* = \tau_w \circ (f_* + dw)$.*

Proof. Since τ_w and ι_X commute and $\nabla_X^F w = (\iota_X \circ d) w$, it is sufficient to prove that $h_*(X) = \tau_w \circ (f_*(X) + \nabla_X^F w)$ holds. Let $m \in M$ and let

$$\gamma\colon (-\varepsilon, \varepsilon) \to M \quad \varepsilon > 0$$

be a curve of class C^∞ with $\gamma(0) = m$ and $\dot{\gamma}(0) = X_m$. We verify the above equation at the point m.

At first we prove the lemma in case when $M = \mathbf{R}^k$. Identifying every tangent space of $\mathbf{R}^k$ to $\mathbf{R}^k$ itself we obtain

$$h_*(X_m) = (h \circ \gamma)^\cdot(0) = \lim_{t\to 0} \frac{1}{t} \{h(\gamma(t)) - h(\gamma(0))\} =$$

$$= \tau_w \lim_{t\to 0} \frac{1}{t} \{f(\gamma(t)) - f(\gamma(0))\} + \tau_w \lim_{t\to 0} \frac{1}{t} \{w_{\gamma(t)} - w_{\gamma(0)}\} =$$

$$= \tau_w \circ (f \circ \gamma)^\cdot(0) + \tau_w \circ \nabla^{\mathbf{R}^k}_{(f\circ\gamma)^\cdot(0)} w = \tau_w \circ (f_*(X_m) + \nabla^{\mathbf{R}^k}_{f_*(X_m)} w) =$$

$$= \tau_w \circ (f_*(X_m) + \nabla^F_{X_m} w).$$

Now let M' be an arbitrary flat manifold and choose a simply connected open neighbourhood U of $f(m)$. Then there exists an isomerty $\varphi\colon U \to \mathbf{R}^k$.

At first we investigate the case when w is small at $m \in M$ with respect to U, i.e. there exists a neighbourhood V of m such that $\exp_{f(m')}(tw_{m'}) \in U$ holds for every $m' \in V$ and $0 \leqq t \leqq 1$. $\varphi_* w | V$ is a vector field along $\varphi \circ f | V$ and

$$\varphi(h(m)) = \varphi(\exp'_{f(m)} w_m) = \exp^{\mathbf{R}^k}_{\varphi(f(m))}(\varphi_* w)$$

holds. From the first step of our proof it follows that

$$\varphi_* \circ h(X_m) = (\varphi \circ h)_*(X_m) = \tau_{\varphi_* w}((\varphi \circ f)_*(X_m) + \nabla^{\mathbf{R}^k}_{(\varphi\circ f)_*(X_m)}(\varphi_* w)) =$$

$$= \tau_{\varphi_* w}(\varphi_* \circ f_*(X_m) + \varphi_*(\nabla_{f_*(X_m)} w)) = \varphi_* \{\tau_w (f_*(X_m) + \nabla_{f_*(X_m)} w)\}$$

and hence

$$h_*(X_m) = \tau_w \circ (f_*(X_m) + \nabla_{f_*(X_m)} w)$$

holds.

Turning to the general case let w be an arbitrary vector field along f and let $\{(U_\alpha, \varphi_\alpha)\colon \alpha \in \mathcal{I}\}$ be an atlas for M' such that for every $\alpha \in \mathcal{I}$ the mapping $\varphi_\alpha\colon U_\alpha \to \mathbf{R}^k$ is an isometry. Let n be a positive integer such that for every $m \in M$ and $0 \leqq t \leqq 1$

the vector field $\tau_{tw}\left(\frac{1}{n}w\right)$ is small at $m\in M$ with respect to some U_α. Then let us define the functions

$$h^k\colon M\to M',\quad k=0,\dots,n$$

by $h^0=f$ and

$$h^k(m)=\exp_{h^{k-1}(m)}\left(\tau_{\frac{k-1}{n}w}\left(\frac{1}{n}w\right)\right)_m$$

for every $m\in M$. It is easy to see that $h^n=h$. It follows that

$$h^1_*(X_m)=\tau_{\frac{1}{n}w}\left(f_*(X_m)+\nabla_{X_m}\left(\frac{1}{n}w\right)\right)$$

and

$$h^2_*(X_m)=\tau_{\frac{1}{n}w}\left(\frac{1}{n}w\right)\left(h^1_*(X_m)+\nabla_{X_m}\left(\tau_{\frac{1}{n}w}\left(\frac{1}{n}w\right)\right)\right)=$$

$$=\tau_{\frac{1}{n}w}\left(\frac{1}{n}w\right)\left(\tau_{\frac{1}{n}w}f_*(X_m)+\tau_{\frac{1}{n}w}\nabla_{X_m}\left(\frac{1}{n}w\right)+\nabla_{X_m}\left(\tau_{\frac{1}{n}w}\left(\frac{1}{n}w\right)\right)\right)=$$

$$=\tau_{\frac{2}{n}w}\left(f_*(X_m)+\frac{2}{n}\nabla_{X_m}w\right).$$

Hence by induction we have

$$h^k_*(X_m)=\tau_{\frac{k}{n}w}\left(f_*(X_m)+\frac{k}{n}\nabla_{X_m}w\right)$$

for every $k=1,\dots,n$. If $k=n$ then our result is obtained.

Theorem (2.1). *Let M be compact, oriented and let M' be flat, complete Riemannian manifold. Then every mapping $f\colon M\to M'$ of class C^p, $p\geqq 2$, is C^p-homotopic to a harmonic mapping.*

PROOF. Consider the de Rham decomposition

$$f_*=du+\Omega,$$

where $u\in\operatorname{Sec}F$ and Ω is harmonic. Let

$$h\colon M\to M'$$

be defined by

$$h(m)=\exp_{f(m)}(-u_m)\quad m\in M.$$

Then, from Lemma (2.3), it follows that f and h are C^p-homotopic. We show that h is a harmonic mapping. Indeed, from Lemma (2.4) we have

$$\partial h_*=-\operatorname{trace}\{(X,Y)\to\nabla^H_X\iota_Y h_*\}=-\operatorname{trace}\{(X,Y)\to\nabla^H_X\iota_Y\tau_u(f_*-du)\}=$$

$$=-\tau_{-u}\operatorname{trace}\{(X,Y)\to\nabla^F_X\iota_Y(f_*-du)\}=-\tau_{-u}(\operatorname{trace}\{(X,Y)\to\nabla^F_X\iota_Y\Omega\})=$$

$$=-\tau_{-u}(\partial\Omega)=0$$

which accomplishes the proof.

In the proof of Theorem (2.1) the homotopy Φ between the mappings f and h has the property that for every $m\in M$ the curve $\Phi(\cdot, m)$ is a geodesic. Two mappings

$$f, h\colon M \to M'$$

of class C^p, $p\in\mathbf{N}$, are said to be strictly homotopic if there exists a C^p-homotopy

$$\Phi\colon [0, 1]\times M \to M'$$

between them such that

$$\Phi(., m)\colon [0, 1] \to M'$$

is a geodesic for every $m\in M$. The mappings f and h are called geodesically homotopic if there exist mappings $h_0, \dots, h_q\colon M\to M'$ with $h_0=f$ and $h_q=h$ such that for every $i=1, \dots, q$ the mappings h_{i-1} and h_i are strictly homotopic.

Lemma (2.6). *The relations of homotopy and geodesic homotopy are identical to each other.*

Proof. It is sufficient to show that if the mappings

$$f, h\colon M \to M'$$

are homotopic then they are geodesically homotopic too. Let Φ be a homotopy between them. From the compactness of M it follows that there exists a positive number ε such that if $m\in M$ and $t, s\in[0, 1]$ with $|t-s|<\varepsilon$ then the points $\Phi(t, m)$ and $\Phi(s, m)$ can be connected with a unique minimal geodesic. Let $0=t_0<t_1<\dots$ $\dots<t_q=1$ be a partition of $[0, 1]$ such that $\max\{|t_i-t_{i-1}|\colon i=1, \dots, q\}<\varepsilon$. Then the mappings $\Phi(t_{i-1}, .)$ and $\Phi(t_i, .)$ are strictly homotopic for every $i=1, \dots, q$.

The following statement enlightens the connection between homotopic harmonic mappings:

Proposition (2.1). *Let M' be flat and complete Riemannian manifold and let*

$$f, h\colon M \to M'$$

be strictly homotopic harmonic mappings of class C^2. Further let $u\in\operatorname{Sec} F$ be the vector field along f with $h(m)=\exp_{f(m)} u_m$ for every $m\in M$. Then for every $m_0\in M$ there exists a neighbourhood U_0 of m_0 and there exists a vector field X_0 on the set $f(U_0)\subset M'$ such that

$$u|U_0 = X_0\circ f$$

is valid.

Proof. By virtue of Lemma (2.5) we have

$$h_* = \tau_u(f_*+du).$$

Applying the operator ∂ to each side of this equation we have

$$0 = \partial h_* = \partial\tau_u(f_*+du) = \tau_u(\partial f_*+\partial du) = \tau_u\partial du,$$

i.e. we obtain that $du=0$. So $\nabla^F_X u=0$ is valid for every vector field $X\in\mathfrak{X}(M)$. Now let $m_0\in M$ and let $f(U_0)$ be a simply connected neighbourhood of $f(m_0)$.

We state that $u_{m_1}=u_{m_2}$ holds for every $m_1, m_{m_2}\in U_0$ with $f(m_1)=f(m_2)$. To see this let

$$\gamma\colon [0, 1] \to U_0$$

be a curve of class C^∞ such that $\gamma(0)=m_1$ and $\gamma(1)=m_2$. Further let Z be an extension of $\dot{\gamma}$ to M. Then $\nabla_Z^F u=0$, i.e. u is parallel along γ. But $f(\gamma(0))=f(m_1)=$ $=f(m_2)=f(\gamma(1))$ and the loop $f\circ\gamma$ is 0-homotopic because $f(U_0)\subset M'$ is simply connected. The manifold M' is flat and so the parallel displacement depends only upon the homotopy classes of curves. It follows that $u_{m_1}=u_{m_2}$. Now let X_0 be defined on $f(U_0)\subset M'$ by $X_0(f(m))=u_m$, $m\in U_0$. It is clear that $u|U_0=X_0\circ f$ and so our statement is proved.

Theorem (2.2). *Let M be compact, oriented and let M' be flat, complete Riemannian manifold. Further let $f\colon M\to M'$ be a harmonic mapping. If f is homotopic to a constant map then f itself is constant.*

Proof. By virtue of a theorem of Hartman in [3] there exists a homotopy

$$\Phi\colon [0, 1]\times M \to M',$$

with $\Phi(0, \cdot)=f$ and $\Phi(1, \cdot)$ = constant map, such that for every $0\leqq t\leqq 1$ the map $\Phi(t, \cdot)\colon M\to M'$ is harmonic. Hence, by virtue of Lemma (2.6) we may suppose that there exist harmonic mappings $h_0, \dots, h_q\colon M\to M'$ with $f=h_0$ and $h_q =$ = constant map such that for every $s=0, \dots, q-1$ the mappings h_s and h_{s+1} are strictly homotopic by a vector field u_s along h_s.

Let us suppose that h_{s+1} is constant for some $s+1\leqq q$. Then we have

$$(h_s)_* = \tau_{u^s}((h_{s+1})_*+du^s) = \tau_{u^s}(du^s).$$

Applying the operator ∂ to each side of the above equation we obtain that $du^s=0$. So $(h_s)_*=0$, i.e. h_s is a constant map. Thus our theorem is proved.

Corollary (2.2). *On flat, complete Riemannian manifold there is no closed geodesic which is* 0*-homotopic.*

As the isometries $S^1\to S^2$ show, we cannot expect the extension of the above corollary to non-flat manifolds.

III. Mappings into commutative groups

Let M and M' be Riemannian manifolds and let $[M, M']$ denote the set of homotopy classes of continuous mappings of M into M'. If $M'=G$ is a Lie group then $[M, G]$ can be endowed with a group structure from G by means of the pointwise multiplication.

If M_1 and M_2 are Riemannian manifolds and

$$\pi\colon M_1 \to M_2$$

is a continuous mapping then it induces a mapping

$$\pi_*\colon [M_2, M'] \to [M_1, M']$$

defined by

$$\pi_*(f) = f \circ \pi \quad \text{for every} \quad f\colon M_2 \to M'.$$

If $M' = G$ is a Lie group then π_* is easily seen to be a homomorphism of $[M_2, G]$ to $[M_1, G]$.

Proposition (3.1). *If* $\pi, \varrho\colon M_1 \to M_2$ *are homotopic continuous mappings then* $\pi_* = \varrho_*$.

Corollary (3.1). *If* M' *is contractible then* $[M, M']$ *is trivial, i.e. there is only one homotopy class.*

Now let

$$\tau\colon M_1' \to M_2'$$

be a continuous mapping of M_1' to M_2'. τ induces a mapping

$$\tau^*\colon [M, M_1'] \to [M, M_2']$$

defined by

$$\tau^*(f) = \tau \circ f \quad \text{for every} \quad f\colon M \to M_1'.$$

If G_1 and G_2 are Lie groups and $\tau\colon G_1 \to G_2$ is a Lie group homomorphism then τ^* is a homomorphism of $[M, G_1]$ to $[M, G_2]$.

Proposition (3.2). *If* $\tau, \sigma\colon M_1' \to M_2'$ *are homotopic continuous maps then* $\tau^* = \sigma^*$.

Corollary (3.2). *If* G *is contractible then* $[M, G]$ *is trivial. Especially* $[M, \mathbf{R}^k]$ *is trivial.*

Proposition (3.3). *If* G_1 *and* G_2 *are Lie groups then*

$$[M, G_1 \times G_2] = [M, G_1] \times [M, G_2].$$

Now we want to determine the group $[M, G]$ for every commutative group G. We need two lemmas as follows:

Lemma (3.1). *A mapping* $f\colon \mathbf{T}^n \to \mathbf{T}$ *is affine if and only if it is harmonic.*

Proof. Let $E_1, \dots, E_n$ be the canonical coordinate fields of $\mathbf{T}^n$ and let us suppose that $f\colon \mathbf{T}^n \to \mathbf{T}$ is an affine mapping. Then f maps geodesics of $\mathbf{T}^n$ onto geodesics of $\mathbf{T}$, i.e.

$$\nabla_{f_*(E_i)} f_*(E_i) = 0$$

holds for every $i = 1, \dots, n$.

So

$$\partial f_* = -\sum_{i=1}^{n} \nabla^F_{E_i} f_*(E_i) = -\sum_{i=1}^{n} \nabla_{f_*(E_i)} f_*(E_i) = 0,$$

i.e. f is harmonic.

Conversely, let us suppose that $f\colon \mathbf{T}^n \to \mathbf{T}$ is a harmonic mapping. Then, by virtue of a theorem of Lichnérowicz of [5] $f = h \circ J$, where $J\colon \mathbf{T}^n \to B(\mathbf{T}^n)$ is a uniquely determined harmonic mapping of $\mathbf{T}^n$ into its canonical torus $B(\mathbf{T}^n)$ and h: $B(\mathbf{T}^n)\to\mathbf{T}$ is an affine mapping. From $b_1(\mathbf{T}^n)=\dim B(\mathbf{T}^n)$ it follows that $B(\mathbf{T}^n)=\mathbf{T}^n$ and $J=id_{\mathbf{T}^n}$, i.e. $f=h$.

Lemma (3.2). $[\mathbf{T}^n, \mathbf{T}]\cong\mathbf{Z}^n$.

PROOF. Every homotopy class of $[\mathbf{T}^n, \mathbf{T}]$ contains a harmonic mapping and so it is easy to see that $[\mathbf{T}^n, \mathbf{T}]$ is isomorphic to the group of affine mappings of $\mathbf{T}^n$ to $\mathbf{T}$ which sends $0\in\mathbf{T}^n$ into $1\in\mathbf{T}$. But if f: $\mathbf{T}^n\to\mathbf{T}$ is an affine mapping with $f(0)=1$ then there exists $(r_1, \ldots, r_n)\in\mathbf{Z}^n$ such that

$$f(\varphi_1, \ldots, \varphi_n) = \exp\left\{i\sum_{j=1}^{n} r_j\varphi_j\right\}, \quad (\varphi_1, \ldots, \varphi_n)\in\mathbf{T}^n$$

which accomplishes the proof.

Theorem (3.1). *Let M be compact, oriented Riemannian manifold and let G be a commutative Lie group. Then*

$$[M, G] \cong \mathbf{Z}^{b_1(M)\cdot b_1(G)}.$$

PROOF. At first let $G=\mathbf{T}$. Then, by virtue of the factorization theorem of Lichnérowicz, we have $[M, \mathbf{T}]\cong[B(M), \mathbf{T}]$. From $\dim B(M)=b_1(M)$ it follows that $[B(M), \mathbf{T}]\cong\mathbf{Z}^{b_1(M)}$ and so the statement is proved when $G=\mathbf{T}$.

In the general case $G=\mathbf{R}^p\times\mathbf{T}^q, p+q=k,$ and hence

$$[M, G] = [M, \mathbf{R}^p\times\mathbf{T}^q] \cong [M, \mathbf{R}^p]\times[M, \mathbf{T}^q] \cong$$
$$\cong [M, \mathbf{T}^q] \cong [M, \mathbf{T}]^q = \mathbf{Z}^{b_1(M)q} = \mathbf{Z}^{b_1(M)b_1(G)}.$$

Thus our theorem is proved.

As an easy consequence of the above theorem we obtain the well-known theorem of [5] as follows:

Corollary (3.4). *Let M be compact and oriented Riemannian manifold. Then $b_1(M)=0$ if and only if every continuous mapping of M into a torus is homotopic to a constant mapping.*

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